Superposition, fields and Gauss’ law

Look again at the Coulomb’s law formula for the force on $A$ due to $B$.

$$
\vec{F}_{AB} = \frac{1}{4\pi \varepsilon_0} \frac{Q_A Q_B (\vec{r}_A - \vec{r}_B)}{|\vec{r}_A - \vec{r}_B|^3}
$$

It has all the right properties. For example, when both charges have the same sign, the force on $A$ points away from $B$, as it should, while the direction flips whenever you change the sign of either charge. Newton’s 3rd law is contained properly within it too, for if one exchanges $A$ and $B$, the magnitude stays the same and the direction flips, again as it should, to produce the right formula for $\vec{F}_{BA}$.

But there is another thing to notice about the electrostatic force. Observe first that it is linear in (i.e. proportional to) each of the charges. This is a symptom of linearity. It turns out that the basic equations of electromagnetism [four partial differential equations known as Maxwell’s equations] are linear. So for example, if there were three charges instead of two, say $A$, $B$, and $C$, then the force on $A$ due to the combined influence of the other two is just the vector sum of the force due to each one separately, each as predicted by Coulomb’s law.

The figure shows the forces acting on charge $A$ due to the other two charges. Dashed arrows indicate the vector displacements $\vec{r}_A - \vec{r}_B$ and $\vec{r}_A - \vec{r}_C$. The picture assumes all charges are the same sign so that the charges repel one another. The net force is the vector sum of the other two forces. That means one can compute them separately and then add vectorially,

$$
\vec{F}_{\text{net}} = \vec{F}_{AB} + \vec{F}_{AC}
$$

This fact is of great importance.

So we see that the force on a given charge due to other charges is just the sum of the force vectors that each one would cause separately. This is called the principle of linear superposition. The effect on a given charge due to a “superposition” of sources is just the vector sum or “superposition” of the effects on each source separately. You recall that this is the way it is with linear differential equations: The particular integral corresponding to a sum of inhomogeneous terms, or “source” terms, is just the sum of the particular integral due to each one separately.

1 I have two comments here: (1) Nature might have been otherwise. There are other kinds of forces in nature that behave like this. It is a physical fact, not a mathematical
Partly because of the superposition property it is useful to introduce the concept of the electric field. We can split the equation for the force on a given charged particle into two separate equations. Consider the electric force on \( A \) due to several different sources, \( B, C, \ldots \). Thus one introduces the idea of the electric field. The other charges create an electric field, which is a vector function, \( \vec{E}(\vec{r}) \) defined at each point \( \vec{r} \) in space. The particle \( A \) feels a force due to the field \( \vec{E}(\vec{r}_A) \) at its own location. The force is the product of the charge \( Q_A \) times the field.

\[
\vec{F}_{\text{net on } A} = \frac{1}{4\pi \varepsilon_o} \sum \frac{Q_B (\vec{r}_A - \vec{r}_B) + Q_C (\vec{r}_A - \vec{r}_C) + \cdots}{|\vec{r}_A - \vec{r}_B|^3} 
\]

Each charged particle plays a dual role. Namely, each particle is a source of electric field, which each other particle “feels,” and at the same time, each particle “feels” a force due to the electric field caused by the other particles. The particle’s charge becomes a handle by which it can connect to the electric field, both as a source and a recipient. In both roles, the particle’s own charge is a proportionality constant. One can imagine the electric field is a sort of “thing” in space by means of which the charges interact. This is similar, at least as an allegory, to the way two cannon balls placed on a trampoline attract each other by means of making the trampoline sag. The “sag” is the electric field. The field exists somehow in space.

\[
\vec{F}_{\text{electric}} = Q \vec{E}(\vec{r}) .
\]

Thus we have in fact two equations from Coulomb’s law plus linearity, namely the one that tells how a charged particle placed in an electric field experiences a force. The latter is actually the definition of the electric field. Then another equation telling how to calculate the field for a set of charges.

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_o} \sum_{\alpha} \frac{Q_\alpha (\vec{r} - \vec{r}_\alpha)}{|\vec{r} - \vec{r}_\alpha|^3}
\]

The latter is sometimes called Coulomb’s law (for the electric field). Now in reality, since elementary charged particles are so small and there are so many of them, it makes more sense to integrate over a charge density \( \rho (\vec{r}) \) than to sum over each charge.

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one, that nature behaves this way rather than some other way. (2) I myself do not know whether linear superposition can be deduced from the Coulomb’s law equation, or not. In the only other inverse-square force law I know, namely gravity in the Newtonian limit, there is also a superposition principle. Maybe one of you can help me with this.
Thus, suppose there is a charge density function \( \rho (\vec{r}') \) such that, if \( V \) is any volume or region in space, then the total charge \( Q \) contained in \( V \) is the volume integral

\[
Q = \int_V \rho (\vec{r}') \, d^3 x'
\]

The notation is that \( d^3 x' \) stands for the volume element, which in Cartesian coordinates would be \( dx \, dy \, dz \). This is really a triple integral, but one tires of writing three integral signs. The prime is to distinguish the dummy variable of integration over the charge from the field-point coordinates \( \vec{r} \). Then if you think of a little “differential” cube of volume \( d^3 x' \) centered at the point \( \vec{r}' \), then the charge in the box is \( \rho (\vec{r}') d^3 x' \) and formula for the electric field at a point \( \vec{r} \) in space is

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int_V \frac{\rho (\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3 x'
\]

This gives you the electric field in space, provided you know the charge distribution. Of course, life is not usually so simple.

As we shall see (Consult here your text, Marion, which you should now begin to read.) the refinements we have made to Coulomb’s law make it logically equivalent to another equation. This other equation is the first Maxwell equation. It is simpler in form, and thus considered more fundamental. It is ...

\[
\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho
\]

This says that the divergence of the electric field at any point in space equals a constant times the electric charge density at that point.

The meaning of the divergence is most easily seen in terms of fluid flow, rather than electric field. If a fluid (like air for example, a gas) flows so that at each point in space its velocity is given by the vector field \( \vec{v}(\vec{r}) \), then the rate at which the fluid flows out of, or “diverges from,” a small cubic box of volume \( d^3 x \) centered at \( \vec{r} \) is the product \( \nabla \cdot \vec{v}(\vec{r}) \, d^3 x \). In terms of the vector components,

\[
\nabla \cdot \vec{E}(\vec{r}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}
\]

Thus the discussion has come to the point where I feel we had better review some mathematics you learned previously, such as surface integrals and the divergence theorem. From this point on, the course will proceed more often by deducing one equation from another using calculus. Keep in mind that this is the course content.