Elliptic functions as Trigonometry

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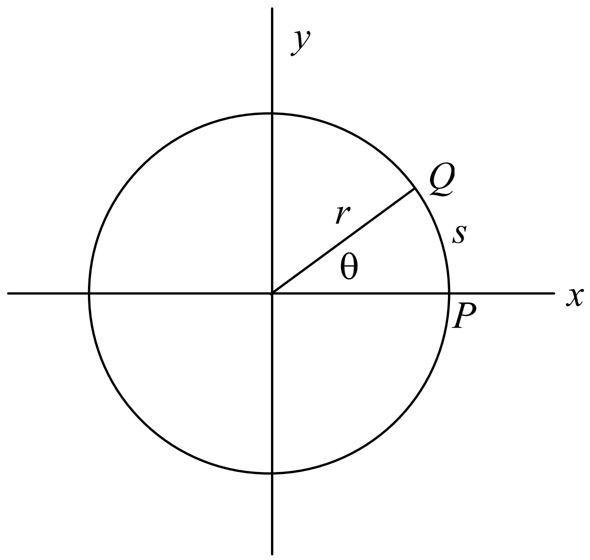


Trigonometry

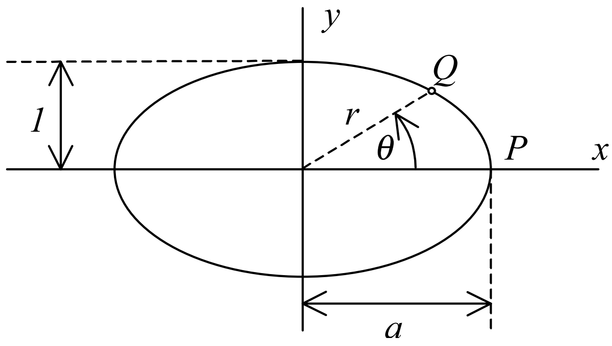
$$x^2 + y^2 = 1, \quad \sin\theta \;=\; \frac{y}{r}, \quad \cos\theta \;=\; \frac{x}{r}.$$

where

$$\theta \;=\; \frac{s}{r} \;=\; \frac{\text{arc length } \overline{PQ}}{\text{radius}}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1, \quad \frac{d}{d\theta}\sin\theta \;=\; \cos\theta, \quad etc.$$

Elliptic functions:



Thus there are *two* relations

$$\frac{x^2}{a^2} + y^2 = 1, \quad \text{(defines ellipse)}$$

$$x^2 + y^2 = r^2, \quad \text{(defines radius)}$$

Elliptic functions

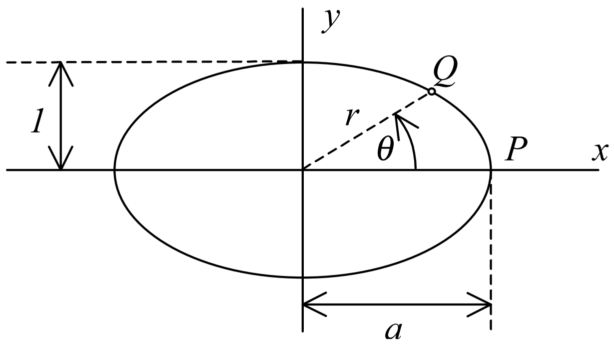
$$\mathrm{sn}(u,k) = y, \quad \mathrm{cn}(u,k) = \frac{x}{a}, \quad \mathrm{dn}(u,k) = \frac{r}{a}.$$

- argument

$$u = \int_P^Q r\, d\theta,$$

- modulus

$$k = \text{eccentricity } \epsilon = \sqrt{1 - \frac{1}{a^2}}$$



$$\frac{x^2}{a^2} + y^2 = 1 \quad \Rightarrow \quad \mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1. \quad (1)$$

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad a^2\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = a^2\,\mathrm{dn}^2(u,k)$$

$$\Rightarrow \quad \mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1. \quad (2)$$

Calculus

Start with $du = r\,d\theta$,

$$\theta = \tan^{-1}\frac{y}{x},$$

so

$$r\,d\theta = r\,\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{x\,dy - y\,dx}{r}.$$

$$du \;=\; \frac{x}{r}\,dy \;-\; \frac{y}{r}\,dx \quad \text{(argument definition)} \tag{3}$$

$$\left.\begin{aligned} dy &= -\tfrac{x}{a^2\,y}\,dx \\ dx &= -\tfrac{y\,a^2}{x}\,dy \end{aligned}\right\} \qquad \text{(ellipse)} \tag{4}$$

Thus for instance, eliminating dx,

$$\begin{aligned} du &= \frac{x}{r}\,dy \;+\; \frac{y}{r}\,\frac{y\,a^2}{x}\,dy \\ &= \left(x \;+\; \frac{a^2\,y^2}{x}\right)\,\frac{dy}{r} \\ &= \left(\frac{x^2}{a^2} \;+\; y^2\right)\,\frac{a^2\,dy}{r\,x}. \end{aligned}$$

Then using the ellipse

$$du \;=\; \frac{a^2}{r\,x}\,dy \quad \text{or} \quad \frac{dy}{du} \;=\; \frac{x}{a}\,\frac{r}{a}.$$

Therefore

$$\frac{d}{du}\,\mathrm{sn}(u,k) \;=\; \mathrm{cn}(u,k)\,\mathrm{dn}(u,k). \tag{5}$$

and similarly

$$\frac{d}{du}\,\mathrm{cn}(u,k) \;=\; -\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k), \tag{6}$$

$$\frac{d}{du}\,\mathrm{dn}(u,k) \;=\; \mathrm{sn}(u,k)\,\mathrm{cn}(u,k). \tag{7}$$

Differential Equations

Elliptic functions satisfy useful nonlinear DEs,

$$\frac{d}{du}\,\mathrm{sn}(u,k) \;=\; \mathrm{cn}(u,k)\,\mathrm{dn}(u,k)$$

$$\left(\frac{d\,\mathrm{sn}(u,k)}{du}\right)^2 \;=\; \Big(1-\mathrm{sn}^2(u,k)\Big)\Big(1-k^2\,\mathrm{sn}^2(u,k)\Big)\,.$$

Therefore $y(x) = \mathrm{sn}(x,k)$ satisfies

$$\left(\frac{d\,y}{d\,x}\right)^2 \;=\; 1 \;-\; (1+k^2)\,y^2 \;+\; k^2\,y^4. \qquad (8)$$

If you differentiate again and cancel $2\,dy/dx$,

$$\frac{d^2\,y}{d\,x^2} \;+\; (1+k^2)\,y \;-\; 2\,k^2\,y^3 \;=\; 0. \qquad (9)$$

(Duffing-type anharmonic oscillator, perihelion shift of Mercury)

Relation to Elliptic Integrals

Equation (8) separates,

$$x \;=\; \int \frac{dy}{\sqrt{(1-y^2)(1-k^2\,y^2)}}, \qquad (10)$$

Fields in waveguides, period of pendulum, Green functions for electrons in two dimensions, electric field of a charged ring, *etc.*