Lesson 1: Selected Exercise Solutions

1. Determine which of the following sentences are propositions.

*d) If $x = 2$, then $x^2 - 2x + 1 = 0$.

Solution: This is not a proposition. As in many examples where a variable is involved, this can be tricky. The truth value of this sentence depends on the value assigned to x. For example, if x is 2, then $x = 2$ is T while $x^2 - 2x + 1 = 0$ is F. So the entire sentence is F. On the other hand, if x is 0, then both $x = 2$ and $x^2 - 2x + 1 = 0$ are F, so the sentence is T. Since the sentence does not have a definite truth value, it is not a proposition. We'll have more to say about this example in the next lesson.

*e) Everybody loves somebody sometime.

Solution: This is a proposition. We can't tell for sure if it T or F (my suspicion is F), but it is certainly one or the other.

2. Construct truth tables for each of the following.

*a) $\neg(p \oplus q)$

Solution:

p	q	$p \oplus q$	$\neg(p \oplus q)$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	T

*d) $\neg(q \vee p) \wedge r$

Solution:

p	q	r	$q \vee p$	$\neg(q \vee p)$	$\neg(q \vee p) \wedge r$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T
F	F	F	F	T	F

3. Perform the indicated bit string operations. The bit strings are given in groups of four bits each for ease of reading.

*b) (1011 1010 $\wedge$ 0110 0110) $\vee$ 0101 0101

Solution:

$$(1011\ 1010 \wedge 0110\ 0110) \vee 0101\ 0101 = 0010\ 0010 \vee 0101\ 0101 = 0111\ 0111$$

4. Let s be the proposition *It is snowing* and f be the proposition *It is below freezing.* Convert the following English sentences into statements using the symbols s, f and logical connectives.

*b) It is below freezing, but it is not snowing.

Solution: $f \wedge \neg s$. Note: In English, *but* often plays the same grammatical role as *and.*

5) Let j be the proposition *Jordan played* and w be the proposition *The Wizards won.* Write the following propositions as English sentences.

*b) $j \longrightarrow w$ *d) $\neg w \longrightarrow \neg j$

Solution:

b) If Jordan played, then the Wizards won.

d) If the Wizards did not win, then Jordan did not play.

7. Use truth tables to verify each of the following equivalences:

*a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Solution:

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$(q \vee r)$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Since the second and fourth columns in the main part of the truth table are identical, the two propositions are equivalent.

*c) $p \vee (p \wedge q) \equiv p$

Solution:

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Since the column for p and the second column in the main part of the table are identical, the two propositions are equivalent.

8. Show that the statements are not logically equivalent.

*b) $(p \longrightarrow q) \not\equiv (q \longrightarrow p)$

p	q	$p \longrightarrow q$	$q \longrightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Since the two columns in the main part of the truth table are not identical, the two propositions are not equivalent.

9. Use truth tables to show that each of the following is a tautology.

*a) $[p \wedge (p \longrightarrow q)] \longrightarrow q$

Solution:

p	q	$p \longrightarrow q$	$p \wedge (p \longrightarrow q)$	$[p \wedge (p \longrightarrow q)] \longrightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the given proposition has truth value T is all cases, it is a tautology.

*c) $(p \wedge q) \longrightarrow p$

<u>Solution</u>:

p	q	$p \wedge q$	$(p \wedge q) \longrightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since the given proposition has truth value T is all cases, it is a tautology.

10. Give proofs of the following equivalences following the pattern at the end of this lesson.

*b) $(p \wedge \neg r) \longrightarrow \neg q \equiv p \longrightarrow (q \longrightarrow r)$

$$\begin{aligned}
(p \wedge \neg r) \longrightarrow \neg q &\equiv \neg(p \wedge \neg r) \vee \neg q && \text{Since } s \longrightarrow t \equiv \neg s \vee t \\
&\equiv (\neg p \vee \neg(\neg r)) \vee \neg q && \text{De Morgan's Law} \\
&\equiv (\neg p \vee r) \vee \neg q && \text{Double Negation Law} \\
&\equiv \neg p \vee (r \vee \neg q) && \text{Associative Law} \\
&\equiv \neg p \vee (\neg q \vee r) && \text{Commutative Law} \\
&\equiv \neg p \vee (q \longrightarrow r) && \text{Since } s \longrightarrow t \equiv \neg s \vee t \\
&\equiv p \longrightarrow (q \longrightarrow r) && \text{Since } s \longrightarrow t \equiv \neg s \vee t
\end{aligned}$$

Lesson 2: Selected Exercise Solutions

1. Let $P(x) : x^2 \leq 4$. Determine the truth values of the following propositions.

*b) $P(-3)$

Solution: $P(-3)$ is the proposition $(-3)^2 \leq 4$. Since $(-3)^2 = 9$ and $9 \not\leq 4$, the proposition $P(-3)$ is false. It has truth value F.

*c) $\forall x\, ((-1 \leq x \leq 1) \longrightarrow P(x))$

Solution: Let's make the assumption that the domain for x is all real numbers. If x is *any* number between -1 and 1, then x^2 will be between 0 and 1, and so it will certainly be true that $x^2 \leq 4$. That means the implication is true for all x in its domain.

So $\forall x\, ((-1 \leq x \leq 1) \longrightarrow P(x))$ it T.

2. Let $P(x, y)$ be x *has been to* y, where the domain of discourse for x is all students in this class, and the domain of discourse for y is all towns in North Dakota. Express the following propositions in English.

*b) $\exists x\, \neg P(x, \text{Hatton})$

Solution: There is a person in this class who has not been to Hatton.

*c) $\exists x\, \forall y\, P(x, y)$

Solution: There is a person in this class who has been to every town in North Dakota.

3. Let $F(x, y)$ be the statement x *can fool* y, where the domain of discourse for both x and y is all people. Use quantifiers to express each of the following statements.

*c) No one can fool everyone.

Solution: $\neg\exists x \forall y F(x, y)$. This proposition is logical equivalent to both $\forall x(\neg\forall y F(x, y))$ and $\forall x \exists y \neg F(x, y)$.

*f) Some people can fool themselves.

Solution: $\exists x F(x, x)$.

*4. Negate each of the statements from exercises 2 in English.

*b) $\exists x\, \neg P(x, \text{Hatton})$

Solution: Everyone in this class has been to Hatton.

*c) $\exists x\, \forall y\, P(x, y)$

Solution: No one in this class has been to every town in North Dakota.

*5. Negate each statement from exercise 3 in logical symbols. Of course, the easy answer would be to simply put $\neg$ in front of each statement. But use the principle given in this lesson to move the negation across the quantifiers.

*c) No one can fool everyone.

Solution: The original (c) could be symbolized as $\neg\exists x\forall y F(x, y)$.
So its negation is $\neg\neg\exists x\forall y F(x, y) \equiv \exists x\forall y F(x, y)$.

*f) Some people can fool themselves.

Solution: The original (f) could be symbolized by $\exists x F(x, x)$.
So its negation is $\neg\exists x F(x, x) \equiv \forall x \neg F(x, x)$.

*6. Express symbolically: *The sum of an even integer and an odd integer is odd.*

Solution: Let's set $E(x)$ to be x *is even*, and set $O(x)$ to be x *is odd*. We'll use the domain for the variable to be all integers in each predicate. The proposition is symbolized by $\forall x\forall y((E(x) \wedge O(y)) \longrightarrow O(x + y))$.

*8. Show $p \vee q$ *and* $\neg p \vee r$, $\therefore q \vee r$ is a valid rule of inference. It is called **Resolution**.

Solution: The easiest way to do this would be to use a truth table to check that the truth values for $((p \vee q) \wedge (\neg p \vee r)) \longrightarrow (q \vee r)$ are always T.

p	q	r	$p \vee q$	$(\neg p \vee r)$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \longrightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	F	F	T

An alternative method would be to use known logical equivalences to provide a logical proof of this rule of inference. That could go as follows:

Argument:

$$
\begin{array}{l}
p \vee q \\
\neg p \vee r \\
\hline
\therefore q \vee r
\end{array}
$$

Proof:

1) $p \vee q$	hypothesis
2) $q \vee p$	commutative law
3) $\neg q \longrightarrow p$	logical equivalence of (2)
4) $\neg p \vee r$	hypothesis
5) $p \rightarrow r$	logical equivalence of (4)
6) $\neg q \longrightarrow r$	hypothetical syllogism using (3) and (5)
7) $\neg\neg q \vee r$	logical equivalence of (6)
8) $q \vee r$	double negation law

*10. Prove

$$\begin{array}{l} \neg p \wedge q \\ r \longrightarrow p \\ \neg r \longrightarrow s \\ s \longrightarrow t \\ \hline \therefore t \end{array}$$

Proof:

1) $\neg p \wedge q$	hypothesis
2) $\neg p$	simplification law from (1)
3) $r \longrightarrow p$	hypothesis
4) $\neg r$	modus tollens from (2) and (3)
5) $\neg r \rightarrow s$	hypothesis
6) s	modus ponens from (4) and (5)
7) $s \longrightarrow t$	hypothesis
8) t	modus ponens from (6) and (7)

*13. Prove the following argument is valid. *All Porsche owners are speeders. No owners of sedans buy premium fuel. Car owners that do not buy premium fuel never speed. So Porsche owners do not own sedans.* Use **all car owners** as the domain of discourse.

Solution: Let's set

$P(x)$: *x owns a Porsche.*

$S(x)$: *x owns a sedan.*

$F(x)$: *x buys premium fuel.*

and

$T(x)$: *x speeds.* (I picked T for *ticket* since S was already used for *sedan*!.)

The argument we need to verify is, in symbols:

Argument:

$$\begin{array}{l} \forall x(P(x) \to T(x)) \\ \neg\exists x(S(x) \wedge F(x)) \\ \forall x(\neg F(x) \longrightarrow \neg T(x)) \\ \hline \therefore \forall x(P(x) \longrightarrow \neg S(x)) \end{array}$$

Proof:

1) $\forall x(P(x) \longrightarrow T(x))$	hypothesis
2) $P(c) \longrightarrow T(c)$ for all car owners c	universal Instantiation (1)
3) $\forall x(\neg F(x) \longrightarrow \neg T(x))$	hypothesis
4) $\neg F(c) \longrightarrow \neg T(c)$ for all car owners c	universal instantiation (3)
5) $T(c) \longrightarrow F(c)$ for all car owners c	logical equivalence of (4)
6) $P(c) \longrightarrow F(c)$ for all car owners c	hypothetical syllogism from (2) and (5)
7) $\neg\exists x(S(x) \wedge F(x))$	hypothesis
8) $\forall x \neg(S(x) \wedge F(x))$	logical equivalence of (7)
9) $\neg(S(c) \wedge F(c))$ for all car owners c	universal instantiation
10) $\neg S(c) \vee \neg F(c)$ for all car owners c	De Morgan's law (9)
11) $\neg F(c) \vee \neg S(c)$ for all car owners c	commutative law
12) $F(c) \longrightarrow \neg S(c)$ for all car owners c	logical equivalence of (11)
13) $P(c) \longrightarrow \neg S(c)$ for all car owners c	hypothetical syllogism from (6) and (12)
14) $\forall x(P(x) \longrightarrow \neg S(x))$	universal generalization

Whew!!

Lesson 3: Selected Exercise Solutions

1. List the members of the following sets.

*b) $\{x \in \mathbb{R} | x^4 = 16\}$

Solution: There are only two real numbers with a fourth power equal to 16, namely 2 and -2, so $\{x \in \mathbb{R}|x^4 = 16\} = \{ 2, -2 \}$.

2. Use set-builder notation to give a description of each set.

*b) $\{1, 2, 3, 4, 5, 6\}$

Solution: There are many possible answers to this question.
The most natural is $\{ n \,|\, n$ is an integer and $1 \leq n \leq 6 \}$.

*3. Determine the cardinality of the sets in exercises 1 and 2.

Solution:

(1b): The cardinality of $\{ 2, -2 \}$ is 2. Or, more compactly, $|\{ 2, -2 \}| = 2$.

(2b): $|\{1, 2, 3, 4, 5, 6\}| = 6$.

*7. Determine the sets A and B, if $A - B = \{1, 5, 7, 8\}, B - A = \{2, 10\}$ and $A \cap B = \{3, 6, 9\}$.

Solution: Since $A - B = \{1, 5, 7, 8\}$, it must be that A has these four integers as elements, and possibly some others. Since $B - A = \{2, 10\}$, B has 2 and 10 as elements and possible some others. So far, we can be sure that $A = \{1, 5, 7, 8, ???\}$ and $B = \{2, 10, ???\}$. Since $A \cap B = \{3, 6, 9\}$ we know these three numbers must be in both A and B, so $A = \{1, 5, 7, 8, 3, 6, 9, ???\}$ and $B = \{2, 10, 3, 6, 9, ???\}$. If there are any elements of A besides $1, 5, 7, 8, 3, 6, 9$, they would also have to be elements of B since otherwise $A - B$ would have elements besides $1, 5, 7, 8$. But then such other elements would belong to $A \cap B$, and that's not true. So we conclude $A = \{1, 5, 7, 8, 3, 6, 9\}$ and $B = \{2, 10, 3, 6, 9\}$.

Alternate Solution: We can let set algebra do the thinking for us: $A = (A - B) \cup (A \cap B) = \{1, 5, 7, 8\} \cup \{3, 6, 9\} = \{1, 5, 7, 8, 3, 6, 9\}$ and $B = (B - A) \cup (A \cap B) = \{2, 10\} \cup \{3, 6, 9\} = \{2, 10, 3, 6, 9\}$.

*8. Use membership tables to show that $A \oplus B = (A \cup B) - (A \cap B)$.

A	B	$A \oplus B$	$A \cup B$	$A \cap B$	$(A \cup B) - (A \cap B)$
1	1	0	1	1	0
1	0	1	1	0	1
0	1	1	1	0	1
0	0	0	0	0	0

Since the third and sixth columns match, the claimed equality is correct.

11. Let $A = \{1,2,3,4\}, B = \{a,b,c\}, C = \{\alpha,\beta\}$, and $D = \{7,8,9\}$. Write out the following Cartesian products.

*c) $C \times B \times D$

Solution:

$$C \times B \times D =$$

$$\{(\alpha,a,7),(\alpha,a,8),(\alpha,a,9),$$

$$(\alpha,b,7),(\alpha,b,8),(\alpha,b,9),$$

$$(\alpha,c,7),(\alpha,c,8),(\alpha,c,9),$$

$$(\beta,a,7),(\beta,a,8),(\beta,a,9),$$

$$(\beta,b,7),(\beta,b,8),(\beta,b,9),$$

$$(\beta,c,7),((\beta,c,8),(\beta,c,9)\}$$

*12. What can you conclude about A and B if $A \times B = B \times A$.

Solution: If $A = \emptyset$ then, no matter what B is, $A \times B = B \times A$ since $A \times B$ and $B \times A$ are both the empty set. Likewise, if $B = \emptyset$, then $A \times B = B \times A$ is certain to be true.

If neither A not B is the empty set, and $A \times B = B \times A$, then it must be that $A = B$. To see why, suppose $a \in A$. Let b be any element of B. Then $(a,b) \in A \times B$. Since $A \times B = B \times A$, we can conclude $(a,b) \in B \times A$, and so $a \in B$. That shows every element of A is also an element of B. Reversing the roles of A and B in the last two sentences shows every element of B is also an element of A. So we have shown $A = B$.

So the punch line is: If $A \times B = B \times A$, then either $A = \emptyset$, or $B = \emptyset$, or $A = B$.

Lesson 4: Selected Exercise Solutions

*2. Give an indirect proof that if the square of the integer n is odd, then n is odd.

The Plan: Before giving the proof, let's analyze what we have to do. First, let $O(n)$ be the predicate *n is odd.* As stated above, the fact to prove is $\forall n(O(n^2) \longrightarrow O(n))$. So to given an indirect proof we'll need to prove $\forall n(\neg O(n) \longrightarrow \neg O(n^2))$. That's means the proof should begin with *Suppose n is not odd* and it should end with *So, n^2 is not odd.* Of course, if you are reading or writing such a proof, this sort of preliminary mental gymnastics is left unmentioned. Instead, the proof is just written as follows:

Proof: Suppose n is an integer which is not odd. Then n is even. So $n = 2k$ for an integer k since to say an integer is even means it is 2 times some integer. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so n^2 is even. That means n^2 is not odd. ♣

*4. Give a proof by contradiction that $\sqrt[3]{2}$ is irrational.

The Plan: To prove this statement by contradiction, we begin by supposing it is false, and show that it is possible to arrive a statement known to be false.

Proof: Suppose $\sqrt[3]{2}$ is rational. That means we can write $\sqrt[3]{2} = \frac{m}{n}$, where m, n are integers, and the fraction is in lowest terms. Cubing both sides gives $2 = \frac{m^3}{n^3}$ which can be rearranged as $2n^3 = m^3$. This shows m^3 is even. Now, using the result in exercise 3, we can conclude m is even. Say $m = 2k$. That means $2n^3 = m^3 = (2k)^2 = 8k^3$, and so $n^3 = 4k^3 = 2(2k^3)$, which shows n^3 is even. That means n must be even. So we have reached a contradiction: The fraction $\frac{m}{n}$ is in lowest terms, and it is not in lowest terms (since m, n have a common factor of 2). $\rightarrow\!\leftarrow$

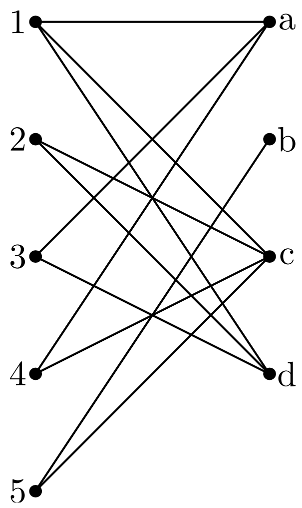
*8. In Lesson 1, exercise 1d, you concluded that *If $x = 2$, then* $x^2 - 2x + 1 = 0$ is not a proposition. Using the convention given in this lesson, what would you say now, and why?

Solution: The convention described in this lesson is that there is an implied universal quantifier preceding the sentence, left for the reader to mentally fill in. So the intended meaning is $\forall x(\text{ if } x = 2, \text{ then } x^2 - 2x + 1 = 0)$ (where it is understood that the domain of x is say all real numbers). Since the variable is now quantified, this sentence is a proposition. In fact it is a false proposition since when the variable has value 2, the $x = 2$ clause is T while the $x^2 - 2x + 1$ clause is F.

Lesson 5: Selected Exercise Solutions

*1. The relation S from $A = \{1, 2, 3, 4, 5\}$ to $B = \{a, b, c, d\}$ is given by the following set of ordered pairs: $S = \{(1, a), (3, a), (4, a), (5, b), (1, c), (2, c), (4, c), (5, c), (1, d), (2, d), (3, d)\}$. Represent S as a bipartite graph.

Solution:



*5. Define a relation on $\{1, 2, 3\}$ which is both symmetric and antisymmetric. You can define your relation either verbally, as a set of ordered pairs, or by drawing a digraph.

Solution: Here is one possible answer as a set of ordered pairs: $R = \{(1, 1), (3, 3)\}$. There are exactly seven other correct answers.

*7. Define the relation $S(A, B)$: A *is a subset of* B, where the domains for A and B are all subsets of $\mathbb{Z}$. Which properties does the relation S satisfy?

Solution:

S is reflexive since every set is a subset of itself. So $A\,S\,A$ for every subset A of $\mathbb{Z}$.

S is not irreflexive since, for just one example, $\{1\}\,S\,\{1\}$.

S is not symmetric since, for just one example, $\{1\}\,S\,\{1, 2\}$, but $\{1, 2\}\,S\,\{1\}$ is false.

S is antisymmetric since for any subsets of $\mathbb{Z}$, if $A\,S\,B$, and $B\ S\,A$, then $A \subseteq B$ and $B \subseteq A$,

so $A = B$.

S is transitive since $A\,S\,B$ and $B\,S\,C$ means $A \subseteq B$ and $B \subseteq C$, so $A \subseteq C$ (as we observed in lesson 3), so that $A\,S\,C$.

*9. Explain why $\emptyset$ (the empty set) is a relation.

Solution: Here is a verbal description of a relation on the set $A = \{ 1 \}$ which is given by $\emptyset$ as a set of ordered pairs: $N(x, y)$: *x is not equal to y.* So $\emptyset$ does represent a relation.

As an alternate solution: $\emptyset$ is a relation since every element of $\emptyset$ is an ordered pair. If you do not believe that, I challenge you to show me an element of $\emptyset$ that is not an ordered pair!

10.

*a) Let $A = \{1\}$, and consider the empty relation, $\emptyset$, on A. Which properties does $\emptyset$ satisfy?

Solution: As usual, it takes some hard thinking to answer these sorts of questions about the empty set!

For the relation $\emptyset$ on the given set A:

$\emptyset$ is not reflexive since $(1, 1)$ is not in $\emptyset$.

$\emptyset$ is irreflexive since we can't find a case of $a, b \in A$ with $a \neq b$ but both $(a, b), (b, a) \in \emptyset$.

$\emptyset$ is symmetric since *If* $(a, b) \in \emptyset \longrightarrow (b, a) \in \emptyset$ is a true proposition. After all, the antecedent $(a, b) \in \emptyset$ is always false, so the implication is true!

The antisymmetric and transitive conditions are also true by exactly the same sort of underhanded reasoning:

$\emptyset$ is antisymmetric since $(a, b), (b, a) \in \emptyset \longrightarrow a = b$ is true.

$\emptyset$ is transitive since $(a, b), (b, c) \in \emptyset \longrightarrow (a, c) \in \emptyset$ is true.

Lesson 6: Selected Exercise Solutions

1. Let A be the set of people alive on earth. For each relation defined below, determine if it is an equivalence relation on A. If it is, describe the equivalence classes. If it is not, determine which properties of an equivalence relation fail.

*c) $a\,L\,b \iff a$ and b have the same last name.

Solution: Let's make the reasonable assumption that this relation is defined on the set of people (and let's ignore the question of what to do with people from societies where last names are not used!).

This relation is an equivalence relation. It is reflexive since every person has the same last name as himself. It is symmetric since if a and b have the same last name, do b and a. Finally, if a and b have the same last name and b and c have the same last name, then a and c have the same last name. So all three requirements for an equivalence relation are met.

It isn't part of the question, but just for practice: The equivalence class of Bill Gates is the set of all people with last name Gates.

*e) $a\,W\,b \iff a$ and b were born less than a day apart.

Solution: This relation is easily seen to be reflexive and symmetric. However, it is not an equivalence relation since it is not transitive. For example, imagine Cal is born 20 hours after Al, and Sal 20 hours after Cal. The $Al\,W\,Cal$ and $Cal\,W\,Sal$ are both true, but $Al\,W\,Sal$ is false.

*4. Complete the proof of the theorem on page 3 of this lesson by proving part (1).

Solution: Suppose E is an equivalence relation on a set A. We need to show that if $a \in [b]$ and $c \in [a]$, then $c \in [b]$. Since $c \in [a]$ we know cEa is true. Since $a \in [b]$, we know aEb is true. Putting cEa and aEb together using the transitivity law, we get cEb. That means $c \in [b]$ as we needed to show. ♣

5. Let $A = \{1, 2, 3, 4, 5, 6\}$. In each case, give an example of a function $f : A \longrightarrow A$ with the indicated properties, or explain why no such function exists.

*b) f is neither one-to-one nor onto.

Solution: Here's one of many possible examples:

$$f(1) = 1 \quad f(2) = 1 \quad f(3) = 3 \quad f(4) = 4 \quad f(5) = 5 \quad \text{and } f(6) = 6$$

*d) f is onto, but not one-to-one.

Solution: There cannot be such a function. If the function is onto, it cannot repeat any values since there are six different inputs and there has to be six different outputs to make the function onto.

*6. Repeat exercise 6, with the set $A = \mathbb{N}$.

*b) f is neither one-to-one nor onto.

Solution: One of many possible answers is $f(n) = 1$ for all $n \in \mathbb{N}$.

*d) f is onto, but not one-to-one.

Solution: One of many possible answers is $f(n) =$ smallest integer bigger than or equal to $\frac{n}{2}$. For example, $f(3) = 2$, $f(4) = 2$, and $f(25) = 13$.

*8. Suppose $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are both one-to-one. Prove $f \circ g$ is one-to-one.

Solution: To show $f \circ g$ is one-to-one, we will suppose $f \circ g(a) = f \circ g(b)$, and show that $a = b$. Now the assumption $f \circ g(a) = f \circ g(b)$ means $f(g(a) = f(g(b))$. Since f is one-to-one, we can conclude $g(a) = g(b)$. But now since g is one-to-one, we get $a = b$ as we needed to prove. ♣

Lesson 8: Selected Exercise Solutions

*2. Draw a (college algebra) graph of $f(x) = \lfloor 2x - 1 \rfloor$.

Hint: Let's think about the function before trying to graph it. When will $f(x) = 0$? That is, when will $\lfloor 2x - 1 \rfloor = 0$? Answer: The result is 0 provided $0 \le 2x - 1 < 1$. That is, provided $\frac{1}{2} \le x < 1$. So the graph will be a horizontal line at $y = 0$ for $\frac{1}{2} \le x < 1$. Next, when will $f(x) = 1$? Answer: When $1 \le 2x - 1 < 2$. In other words, for $1 \le x < \frac{3}{2}$ the graph will be a horizontal line at $y = 1$. Continue this sort of reasoning, and you will be able to draw the graph.

*4. Let $f(x) = 4x^5$ and let $g(x) = 2^x$. Does g ever catch up with f, or does f always stay ahead of g?

Solution: g does eventually catch f and stays ahead of it after that. This may seem hard to believe if you just experiment with small values of x. For example,

$f(1) = 4$ while $g(1) = 2$

$f(2) = 128$ while $g(2) = 4$

$f(3) = 972$ while $g(2) = 8$

$f(10) = 400000$ while $g(2) = 1024$

and at $x = 20$, the value of f is 11751424 ahead of the value of g.

But when x get very large, the effect of the x in the exponent in 2^x will overwhelm the effect of the base x in $4x^5$.

For example,

$f(25) = 39062500$ while $g(25) = 33554432$ so now f is ahead by a mere 5508068, and g is closing the gap!

At $x = 25.3$, f is ahead by only 152828, and at $x = 25.4$, g has passed f and is now ahead by .00000198!

The gap between f and g only gets larger for larger values of x. For example, at $x = 26$, g is ahead by 19583360.

With a few more mathematical tools, it's possible to determine where g catches f (it's at about $x = 25.3075$), also to prove g stays ahead of f after that. But only the sort of numerical experimenting as given above is expected as an answer to this problem.

*5. The x^y button on you calculator is broken. Show how can you approximate $2^{\sqrt{2}}$ with your calculator anyhow.

Solution: There are many possible answers. Here's one: We need to calculate $t = 2^{\sqrt{2}}$. Taking logs of both sides gives $\log t = \log 2^{\sqrt{2}} = \sqrt{2}\log 2$. That last quantity we can compute with the calculator, getting $\log t = .4257\cdots$. We can now just hit the 10^x button since $10^{\log t} = t$. So $2^{\sqrt{2}} = t = 10^{.4257\cdots} = 2.6651\cdots$.

*6. What is the next term in the sequence $1, 2, 3, 5, 6, 7, \cdots$? What's another possible answer?

Solution: These sorts of questions are really a little silly taken at face value. Absolutely any number is a possible next term in the sequence. There is always a formula or rule that will justify any value you want. So maybe a better way to state the problem is: I am thinking of a *simple* rule used to build a sequence and the first six terms are $1, 2, 3, 5, 6, 7$. Guess the rule!

One possible answer would be that the rule is to list positive integers in increasing order that are not multiples of 4. So the sequence would run $1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, \cdots$,

Another possible rule: list the positive integers in increasing order that do not have the number of letters in their (English) name equal to the number itself. So 1 is included since *one* is not a word with one letter. In fact, 4 is (probably) the only number not listed. So the sequence would run $1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, \cdots$

One more solution: If we count Sunday as day 1, Monday as day 2, and so on, until Saturday is day 7, then starting in 2006, the day of the week January 1^{st} falls on is given by the sequence $1, 2, 3, 5, 6, 7, 1, 3, 4, 5, 6, 1, \cdots$

*9. Evaluate $\sum_{j=1}^{4} j^2$.

Solution: Since there are only four terms in the sum, we can do this by *brute force*. Of course, if there were 4000 terms in the sum, we would look for a better way!

$$\sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$$

*12. What is the sum of the first twenty terms of the geometric sequence with initial term 3 and common ratio 2?

Solution: This one is almost still small enough to sum by brute force, but let's use the geometric sum formula instead:

The sum we are asked to compute is

$$3+3(2)+3(2^2)+3(2^3)+\cdots 3(2^{19}) = \sum_{j=0}^{19} 3(2^j) = 3\sum_{j=0}^{19} 2^j = 3\left(\frac{2^{20}-1}{2-1}\right) = 3(2^{20}-1) = 3145725$$

Lesson 8: Selected Exercise Solutions

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Lesson 9: Selected Exercise Solutions

*2. List the first five terms of the sequence defined recursively by $a_0 = 0$, and, for $n \geq 1$, $a_n = 3a_{n-1} + 1$. Guess a closed form formula for the sequence. (Hint: It might help if you double the terms of the sequence.)

Solution: The sequence begins $0, 1, 4, 13, 40$. Taking the hint, doubling these we get $0, 2, 8, 26, 80$ and now we recognize these as 1 less than powers of 3: $1, 3, 9, 27, 81$. So it looks like a reasonable guess is that $2a_n + 1 = 3^n$, or $a_n = \dfrac{3^n - 1}{2}$.

*4. Let r be a fixed real number different from 0. For a positive integer n, the symbol r^n means the product of n r's. For convenience, r^0 is defined to be 1. Give a recursive definition of r^n analogous to the definition of $n!$ given in this chapter.

Solution: $r^0 = 1$. For $n \geq 1$, $r^n = r(r^{n-1})$.

*7. Give a recursive definition of the set of positive integers that are not multiples of 4.

Solution: There are many possible answers to this problem. If we call the set S, probably the most natural description of S is (1) $1, 2, 3 \in S$, and (2) If $n \in S$, then $n + 4 \in S$.

*9. Describe the strings in the set S of strings over the alphabet $\Sigma = \{a, b, c\}$ defined recursively by (1) $\lambda \in S$ and (2) if $x \in S$, then $abcx \in S$.

Solution: Let's use the rules to build elements of S until we can guess a description of the elements in S: $\lambda, abc, abcabc, abcabcabc, abcabcabcabc, \cdots$. OK, the answer is clear: S consists of strings made up of 0 or more repetitions of the string abc.

*11. A **palindrome** is a string that reads the same in both directions. For example, *aabaa* is a palindrome of length five and *babccbab* is a palindrome of length eight. Give a recursive definition of the set of palindromes over the alphabet $\Sigma = \{a, b, c\}$.

Solution: Let's call the set of palindromes P. The plan is to list the shortest elements of P, and then give a rule for making larger palindromes from smaller ones. Here's one possible solution: (1) λ, a, b, $c \in P$ and (2) if $x \in P$ then axa, bxb, $cxc \in P$.

Notice that we had to include a, b, $c \in P$ in rule (1) since starting with just $\lambda \in P$ and applying rule (2), we will not be able to construct the palindromes a, b and c (or any other odd length palindromes for that matter).

Here is a second, less obvious, way to describe P: (1)λ, a, b, $c \in P$ and (2) if $x, y \in P$, then $xyx \in P$. In plain English, to build longer palindromes, take any palindrome and stick a palindrome on each end! Experiment with this definition a bit to see how it works. One nice thing about this second definition is that there is a single *building* rule instead of three different rules as in the first description given above.

Lesson 10: Selected Exercise Solutions

*2. Prove: For every integer $n \geq 1$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Solution: For the basis case, we make sure $\frac{1}{1\cdot 2}=\frac{1}{1+1}$, and that is true since both quantities equal $\frac{1}{2}$.

For the inductive step, suppose

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for some integer $n \geq 1$. Then, adding $\frac{1}{(n+1)(n+1+1)}$ to the left side we get,

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}\right]+\frac{1}{(n+1)(n+1+1)}$$

$$=\left[\frac{n}{n+1}\right]+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}$$

$$=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}$$

as we needed to show.

*4. Prove: For every integer $n > 4$, $2^n > n^2$.

Solution: In this problem, $n = 5$ is the basis case, and since $2^5 = 32 > 25 = 5^2$, the basis for the induction is established.

Suppose now that $2^n > n^2$ for some integer $n > 4$. We need to show $2^{n+1} > (n+1)^2$. So let's compute. The plan is to start with $(n+1)^2$, and make over estimates until we see the quantity is less than 2^n.

$(n+1)^2 = n^2 + 2n + 1 < n^2 + 2n + n$ since $n > 4$

$= n^2 + 3n < n^2 + (n)n$ since, again, $n > 4$

$= n^2 + n^2 = 2n^2 < 2(2^n) = 2^{n+1}$ where we used $n^2 < 2^n$ in the next to last step

*7. A sequence is defined recursively by $a_0 = 0$, and, for $n \geq 1$, $a_n = 5a_{n-1} + 1$. Use induction to prove the closed form formula for a_n is $a_n = \dfrac{5^n - 1}{4}$.

Solution: Checking the basis $n = 0$: $\dfrac{5^0 - 1}{4} = \dfrac{0}{4} = 0$ which does equal a_0. Suppose now that $a_k = \dfrac{5^k - 1}{4}$ for some $k \geq 0$, Then

$$a_{k+1} = 5a_k + 1 = 5\left(\frac{5^k - 1}{4}\right) + 1 = \frac{5^{k+1} - 5}{4} + 1 = \frac{5^{k+1} - 5 + 4}{4} = \frac{5^{k+1} - 1}{4}$$

as we needed to show. ♣

Lesson 11: Selected Exercise Solutions

*1. Consider the following algorithm: The input will be two integers, $m \geq 0$, $n \geq 1$.

step 1: **set** $p = 1$

step 2: **if** $m = 0$, **then output** p and **stop**

step 3: **replace** p by np

step 4: **replace** m by $m - 1$ and **go to** step 2

Describe in words what this algorithm does. In other words, what problem does this algorithm solve?

Solution: Let's experiment with a *typical* choice for m and n to see if we can guess the effect of this algorithm. Set $m = 3$ and $n = 2$.

Here is the trace for these choices:

step 1: $p = 1$

step 2: $m \neq 0$ so continue to step 3.

step 3: $p = (2)(1)$

step 4: $m = 2$, go to step 2

step 2: $m \neq 0$ so continue to step 3.

step 3: $p = (2)(2)(1)$ (let's not do the arithmetic since that might disguise what is going on)

step 4: $m = 1$, go to step 2

step 2: $m \neq 0$ so continue to step 3.

step 3: $p = (2)(2)(2)(1)$

step 4: $m = 0$, go to step 2

step 2: $m = 0$, output 8 (since $(2)(2)(2)(1) = 8$) and stop.

OK, the operation looks like is puts an additions factor of n each time through the loop, and since it does this m the result is that $(n)(n)(n)\cdots(n)$ (with m factors of n) is printed out. In other words, the algorithm computes n^m. Notice that if $m = 0$ initially, the 1 is output, and so even in this case, the correct value of n^m is output.

2. For the algorithm of problem 1:

*(a) Select a value to represent the *size* of an instance of the problem the algorithm is designed to solve.

Solution: Many different answers are possible here. Probably the most natural would be to say the operation in step 3 eats up the majority of the effort involved in applying this algorithm, and since the number of repetitions of that step depends on m, it's natural to let m represent the size of the of an instance of this algorithm.

*5: Design an algorithm that will convert the ordered triple (a, b, c) to the ordered triple (b, c, a). For example, if the input is $(7, X, *)$, the output will be $(X, *, 7)$.

Solution: This is a common task in computer programming: moving data from place to place. The thing to worry about is overwriting part of the data that will be needed later. Here's an algorithm that does the trick:

The input will be a triple of symbols: (a, b, c)

step 1: **set** temp $= a$ (we need to save this value before erasing it!)

step 2: **replace** a with b

step 3: **replace** b with c

step 4: **replace** c with temp (the value of a we saved in step 1)

(So the triple now looks like (b, c, a) as required. We'll output it to prove that!)

step 5: **output** (a, b, c) and **stop**.

Practice with this algorithm until you can see clearly what it is doing. In particular, be sure the (a, b, c) in the step 5 makes sense.

*7. A **palindrome** is a sting of letters that reads the same in each direction. For example, *refer* and *redder* are palindromes of length five and six respectively. Design an algorithm that will take a string as input and output *yes* if the string is a palindrome, and *no* if it is not.

Solution: There are many possible correct answers. Here is one:

The input will be a string s:

step 1: **set** $n =$ number of symbols in the string.

step 2: **if** $n = 0$ of $n = 1$, output *palindrome* and **stop**

step 3: **if** first and last symbols in s are different, output *not a palindrome* and **stop**

step 4: **delete** the first and last symbols from s go to step 1.

Examples:

For input string $s = abba$ the algorithms follows these steps:

step 1: $n = 4$

step 2: n isn't 0 or 1 so go to next step.

step 3: a is the same as a so go to step 4

step 4: $s = bb$

step 1: $n = 2$

step 2: n isn't 0 or 1 so go to next step.

step 3: b is the same as b so go to step 4

step 4: $s = \lambda$, the empty string.

step 1: $n = 0$ so output *palindrome* and stop.

For input string $s = abca$ the algorithm follows these steps:

step 1: $n = 4$

step 2: n isn't 0 or 1 so go to next step.

step 3: a is the same as a so go to step 4

step 4: $s = bc$

step 1: $n = 2$

step 2: n isn't 0 or 1 so go to next step.

step 3: b is the not the same as c so output *not a palindrome* and stop.

Lesson 12: Selected Exercise Solutions

For the proofs requested below, use facts and theorem given in this lesson as justifications. Assume all letters represent integers.

*1. Prove that if $a > 0$ and $b > 0$, then $ab > 0$.

Solution: Suppose the integers a and b are both greater than 0.

Since $a > 0$, multiplying both sides of $0 < b$ by a will give $a0 < ab$ according to one of the rules of inequalities given in the lesson. A theorem proved in this lesson tells us $a0 = 0$. So $a0 < ab$ is the same as $0 < ab$, and that is what we needed to show. ♣

*4. Prove the cancellation law for multiplication: For integers a, b, c, with $c \neq 0$, if $ac = bc$, then $a = b$.

Hint: The fact proved in exercise 3 will be useful.

Solution: It's tempting to say *multiply each side by* $\frac{1}{c}$. but there is nothing in this lesson that permits such a step. We haven't yet officially discussed the idea of fractions! So we'll have to work a little harder.

Suppose $c \neq 0$ and that $ac = bc$. Add $-bc$ to each side of that last equation to get $ac - bc = bc - bc$. Now $bc - bc = (b - b)c = 0c = 0$, so $ac - bc = 0$. Using the distributive law, that last equation can be written as $(a - b)c = 0$. According to exercise 3, either $a - b = 0$ or $c = 0$. Since the second option is definitely wrong, it must be that $a - b = 0$. Adding b to each side of that equation tells us $a - b + b = 0 + b$ so that $a + 0 = b$. In other words, $a = b$. ♣

*6. Prove: For $a, b, c \in \mathbb{Z}$, if $a|b$, then $-a|b$.

Solution: Suppose $a|b$. That means $ac = b$ for some integer c. Then $(-a)(-c) = ac = b$, so $-a|b$. ♣

*8. Determine if 1297 is a prime.

Solution: Since $\sqrt{1297} = 36.01\cdots$, we will need to check to see if 1297 is divisible by $2, 3, 5, 7, 11, 13, 17, 19, 23, 29,$ or 31. Testing those 11 values, we find none divides 1297, and so 1297 is a prime.

Lesson 13: Selected Exercise Solutions

*3. Write a step-by-step algorithm that implements the Euclidean algorithm for finding gcd's.

Solution: gcd algorithm: Input: two natural numbers, a,b, not both 0.

step 1: **if** $b = 0$, **then output** a and **stop**

step 2: divide a by b to find remainder r

step 3: **replace** a by b, and b by r and go to step 1

Remark: Many computer languages have a built in operator to give the remainder when one integer is divided by a second. In the C language the notation is $a\%b$. So, for example $25\%7 = 4$ in that language.

*6. What can you conclude about $gcd(a, b)$ if there are integers s, t with $as + bt = 1$?

Solution: Since the smallest positive integer that is a linear combination of a and b is $gcd(a, b)$, and since there are no positive integers smaller than 1, we can conclude $gcd(a, b) = 1$.

Lesson 15: Selected Exercise Solutions

*2. Determine the prime factorization of 5183.

Solution: The sqruare root of 5183 is certainly less than 80 since $80^2 = 6400$. So, unless 5183 is a prime, we are bound to find a prime factor less than 80. Let's seach for one by brute force. Testing $2, 3, 5, 7, 11 \cdots$, we eventually discover that $71|5183$, and, in fact, $5183 = 71 \cdot 73$.

*3. List all the positive divisors of 5183.

Solution: Since $5183 = 71 \cdot 73$, there will only be four positive divisors of 5183. They are

$$71^0 73^0 = 1 \qquad 71^1 73^0 = 71 \qquad 71^0 73^1 = 73 \qquad \text{and} \qquad 71^1 73^1 = 5183$$

*5. Find all integer solutions to $33x + 12y = 7$.

Solution: Since $gcd(33, 12) = 3$ and 3 does not divide 7, there are no solutions.

*6. Find all integer solutions to $33x + 12y = 6$.

Solution: We need one solution to get the ball rolling. We could use the continued fraction algorithm to write $3 = gcd(33, 12)$ as a linear combination of 33 and 12, and we would probably have to do that if the numbers were larger. But with these small numbers we can do the work in our head: $3 = (33)(-1) + (12)(3)$.

Multiply that equation by 2 on each side to get $(33)(-2) + (12)(6) = 6$.

So now we have one solution to the given equation: $x = -2$ and $y = 6$.

Using the formulas that produce all solutions once one is known we get all solutions are given by

$$x = -2 + \frac{12}{3}k = -2 + 4k \qquad \text{and} \qquad y = 6 - \frac{33}{3}k = 6 - 11k$$

where k is any integer.

For example, when $k = 5$ we get the solution $x = 18$, $y = -49$.

Lesson 16: Selected Exercise Solutions

1.

*(b) What day of the week is it 3122 days after a Monday?

Solution: Since $3122 \equiv 0\,(mod\,7)$, it will again be a Monday.

*4. Determine n between 0 and 19 such that $2311 + 3912 \equiv n$ mod 20.

underbarSolution:

$2311 + 3912 \equiv 11 + 12 \equiv 23 \equiv 3\,(mod\,20)$

*7. Solve: $4x \equiv 1$ mod 7.

Solution: Since 7 is such a small number, we can do this problem by brute force. There are only seven congruences to check:

$$(4)(0) \equiv 0 \not\equiv 1\,(mod\,7)$$

$$(4)(1) \equiv 4 \not\equiv 1\,(mod\,7)$$

$$(4)(2) \equiv 8 \equiv 1\,(mod\,7)$$

$$(4)(3) \equiv 12 \not\equiv 1\,(mod\,7)$$

$$(4)(4) \equiv 16 \not\equiv 1\,(mod\,7)$$

$$(4)(5) \equiv 20 \not\equiv 1\,(mod\,7)$$

$$(4)(6) \equiv 24 \not\equiv 1\,(mod\,7)$$

So the only solution is $x \equiv 2\,(mod\,7)$.

Solving the same problem by the more general method described in the notes we would begin by writing $1 = gcd(7, 4)$ as a linear combination of 7 and 4: $(-1)(7) + (2)(4) = 1$. Of course we could do that one in our heads. For larger values, we would use the continued fraction algorithm to find such a linear combination. From the equation $(-1)(7) + (2)(4) = 1$ we can read off the solution to $4x \equiv 1\,(mod\,7)$ as $x \equiv 2\,(mod\,7)$.

*10. Convert to decimal: 12_3, 123_4, 1234_5, and ABC_{16}.

Solution: $ABC_{16} = (10)(16^2) + (11)(16) + 12 = 2748$

*11. Convert the decimal integer 23109 to bases 2, 6, and 16. Remember to use $A, \cdots, F$ to represent base 16 digits from 10 to 15.

Solution:

$$23109 = (1444)(16) + 5$$
$$1444 = (90)(16) + 4$$
$$90 = (5)(16) + A$$
$$5 = (0)(16) + 5$$

So The decimal number 23109 is $5A45_{16}$ (hex).

*13. Make base 5 addition and multiplication tables similar to the base 7 multiplication table of problem 13.

Solution: Here's the base 5 addition table:

+	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

And here is the base 5 multiplication table:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

Lesson 17: Selected Exercise Solutions

*2. Using the data of problem 1, a student has decided to take one biology, one physics, and one chemistry course. How many different such selections are possible?

Solution: There are 12 choices for the biology course, 10 for the physics course, and 4 for the chemistry course. So, according to the product rule, there are $12 \cdot 10 \cdot 4$ ways to set up a schedule comprised of one course from each department.

*4. How many words of length six are there if letters may be repeated? (Examples: BBBXBB, ABATBC are OK).

Solution: Assuming the letters are from the usual 26 letter alphabet, and that the case does not matter, the product rule says the answer is 26 times itself 6 times: 26^6.

*7. How many binary strings of length less than or equal to nine are there?

Solution: There is one binary string of length 0, 2 of length 1, 2^2 of length 2, $\cdots$, 2^9 of length 9. So, according to the sum rule, there are $1+2+2^2 \cdots + 2^9$ binary strings with lengths nine or less. We could use the geometric sum formula to write the total as $\frac{2^{10}-1}{2-1} = 2^{10}-1 =$ 1023.

*11. In how many ways can those same 26 volumes be placed on a shelf if the volumes labeled with vowels must be adjacent?

Solution: Let's begin by imaging the five volumes labeled **a,e,i,o,u** are bound together with a rubber band (or maybe, more appropriately considering the subject matter, by alien matter-antimatter magnetic ectoplasm). In any case, we will move these five volumes as a unit called V (for vowels). That means we actually have 22 items to arrange on the shelf: **b,c,d,f,g,h,j,k,**$\cdots$**,x,y,z,** and **V**. There are 22! to arrange those 22 items. That completes task #1. Once an arrangement of those 22 items has been selected, we will then apply the reverse antimatter-matter degausser so we can arrange the five vowel volumes among themselves. We can do this task #2 in 5! ways. Since we need to do both tasks, we conclude there are 22! 5! to arrange to books on the shelf.

*12. In how many ways can 6 men and 4 women sit in a row?

Solution: Since there are 10 people with no restrictions on order, there are 10! possible arrangements.

*15. A lottery ticket consists of five different integers selected from 1 to 99. How many different lottery tickets are possible? How many tickets would you need to buy to have a one-in-a-million chance of winning by matching all five randomly selected numbers?

Solution: There are $\binom{99}{5}$ different tickets. So the chances of any one ticket being a winner is $\frac{1}{\binom{99}{5}}$.

So, assuming we buy n different tickets, the chances of winning are $\frac{n}{\binom{99}{5}}$.

We want to pick n so that $\frac{n}{\binom{99}{5}} \geq \frac{1}{1000000}$.

Rearrange that, and we see we need n so that $n \geq \frac{\binom{99}{5}}{1000000}$.

Now $\binom{99}{5} = \frac{99 \cdot 98 \cdot 97 \cdot 96 \cdot 95}{5!} = 71523144$

So we need at least $n \geq \frac{71523144}{1000000} = 71.523144$ tickets.

In other words, we need to buy 72 tickets.

Lesson 18: Selected Exercise Solutions

*4. Give a combinatorial proof that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Hint: How many ways are there to select a 2-subset of $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots b_n\}$?

Solution: We can make a set of size two in three independent ways: Select two a's, select two b's, or select one of each. We can select 2 of the n a's in $\binom{n}{2}$ ways, we can select 2 b's in $\binom{n}{2}$ ways, and, we can select one of each letter in $n \cdot n$ ways. That gives a total of

$$\binom{n}{2} + \binom{n}{2} + n \cdot n = 2\binom{n}{2} + n^2$$

ways to select two elements of that set. On the other hand, since there are $2n$ elements in the set, there are $\binom{2n}{2}$ ways to select two elements from it. The two answers we've found for the question about how many ways are there to select two elements of the set must be equal. So

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

as we needed to show.

*6. Show that if p is a prime and $0 < k < p$, then p divides $\binom{p}{k}$. Hint: When $\binom{p}{k}$ is written out, how many times does p occur as a factor of the numerator and how many times as a factor of the denominator?

Solution: Writing out the binomial coefficient we get

$$\binom{p}{k} = \frac{p \cdot (p-1) \cdot (p-2) \cdots (3)(2)(1)}{k \cdot (k-1) \cdots (2)(1)(p-k)(p-k-1) \cdots (2)(1)}$$

Since $0 < k < p$, the neither of the products $k \cdot (k-1) \cdots (2)(1)$, $(p-k)(p-k-1) \cdots (2)(1)$ contain a factor of p to cancel the p in the numerator. So, when $\binom{p}{k}$ is reduced to an integer, the factor of p will remain. That means p divides $\binom{p}{k}$.

*9. How many positive integers between 1000 and 9999 inclusive are not divisible by any of $4, 10$ or 25 (careful!)?

Solution: Let A be the integers between 1000 and 9999 that are multiples of 4. B will be the integers in that range that are multiples of 10 and C will be those that are multiples of 25. We can answer the question posed by first determining $|A \cup B \cup C|$. By the inclusion-exclusion formula we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Since $1000 = (250)(4)$ and $10000 = (2500)(4)$ we see there are $|A| = 2500 - 250 = 2250$ multiples of 4 between 1000 and 9999 inclusive. Likewise, there are $|B| = 1000 - 100 = 900$ multiples of 10 in that range, and $|C| = 400 - 40 = 360$ multiples of 25.

To count $A \cap B$, notice that saying an integer is divisible by both 4 and 10 is really the same as saying it is divisible by 20 (the least common multiple of 4 and 10). So $|A \cap B| = 500 - 50 = 450$. In the same way $|A \cap C| = 100 - 10 = 90$ and $|B \cap C| = 200 - 20 = 180$. Finally, the integers in $A \cap B \cap C$ are the multiples of 100, and so $|A \cap B \cap C| = 100 - 10 = 90$.

Putting all these numbers together we get

$$|A \cup B \cup C| = 2250 + 900 + 360 - 450 - 90 - 180 + 90 = 2880$$

Since we want to count the number not divisible by any of these, we need the total number of integers in the set minus the number we just counted (*Good* = *Total* - *Bad*). So the answer to the problem is $9000 - 2880 = 6120$

*11. How many permutations of the digits $1, 2, 3, 4, 5$ have no digit in its own spot?

Solution: The answer to problem 10 gives the number of permutations of $1, 2, 3, 4, 5$ with at least one digit in its own spot. In this problems, those would be the *bad* permutations. Since there are 5! total permutations, the *Good* = *Total* - *Bad* rule says the answer is:

$5! -$ *(answer to problem 10).*

Lesson 19: Selected Exercise Solutions

*3. How many cards must be selected from a deck to be sure that at least 5 of the selected cards have the same suit?

Solution: Since there are four suits, the general pigeonhole principle formula tells us the minimum number of cards we need to select is the smallest value of n so that $\lceil \frac{n}{4} \rceil = 5$, and a little experimenting shows that $n = 17$ is the solution.

We can also just use common sense to answer the question: Since there are four suits, would could select 16 cards, four of each suit and so not getting five of any one suit, but the 17^{th} card will then certainly give a fifth card of some suit.

*4. Show that in any set of 217 integers, there must be a pair with a difference that is a multiple of 216.

Solution: Since there are only 216 different values modulo 216, the pigeonhole principle says some two of the 217 numbers, say m and n, must have the same value modulo 216. So $m \equiv n\,(mod\,216)$. That means $216\,|\,m - n$. So m and n have a difference that is a multiple of 216.

*7. A (cheap) vending machine accepts pennies, nickels, and dimes. Let d_n be the number of ways of depositing n cents in the machine, where the order in which the coins are deposited matters. Determine a recurrence relation for d_n. Give the initial conditions.

Solution: For the recurrence relation, use the following reasoning. assuming $n \geq 10$, we can begin to deposit the coins in one of three ways:

1) Deposit a penny. There are d_{n-1} ways to add the remaining coins totaling $n-1$ cents.

2) Deposit a nickle. There are d_{n-5} ways to add the remaining coins totaling $n-5$ cents.

3) Deposit a dime. There are d_{n-10} ways to add the remaining coins totaling $n-10$ cents.

So, by the sum rule, we get $d_n = d_{n-1} + d_{n-5} + d_{n-10}$.

For initial conditions, we will need to determine $d_0, d_1, \cdots, d_9$.

Actually, for these values, we can only use pennies and nickles, and the example in the text tells us these values is given by

$$d_0 = 1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 1, \quad \text{and for} \quad 5 \leq n \leq 9, \qquad d_n = d_{n-1} + d_{n-5}$$

So the initial conditions for our problem are

$$d_0 = 1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 1,$$

$$d_5 = 2, \quad d_6 = 3, \quad d_7 = 4, \quad d_8 = 5, \quad \text{, and} \quad d_9 = 6$$

*8. Al climbs stairs by taking either one or two steps at a time. For example, he can climb a flight of three steps in three different ways: (1) one step, one step, one step or (2) two step, one step, or (3) one step, two step. Determine a recursive formula for the number of different ways Al can climb a flight of n steps.

Solution: Let c_n be the number of ways to climb a flight of n steps. If Al first takes a one step, he can complete the climb in c_{n-1} ways. If he starts with a two step, he can complete the climb in c_{n-2} ways. By the sum rule, we conclude $c_n = c_{n-1} + c_{n-2}$.

For the initial conditions: $c_0 = 1$ and $c_1 = 1$.

*11. Find a recurrence relation for the number of bit strings of length n that contain two consecutive 0's.

Solution: Let g_n be the number of *good* bit strings of length n (*good* meaning the string contains two consecutive 0's).

Here's a neat way to think about the recursive relation: A *good* string of length n (where $n \geq 2$) will end with either a 0 or a 1. If it ends with a 1, the preceding $n-1$ bits must be a *good* string. If it ends with a 0, there are two possibilities: it could end with 10, in which case the preceding $n-2$ bits must be a *good* string, but if it ends 00, then any string of length $n-2$ can precede it. So by the sum rule we get

$$g_n = g_{n-1} + g_{n-2} + 2^{n-2} \qquad \text{for } n \geq 2$$

To complete the problem we give the initial conditions: $g_0 = 0$ and $g_1 = 0$.

*12. Find a recurrence relation for the number of bit strings of length n that contain the string 01.

Solution: Let g_n be the number of *good* bit strings of length n (*good* meaning the string contains the pattern 01). We can give a non-recursive formula for g_n since it is easy to see that the *bad* strings of length n are the strings made up of 0 or more 1's followed by enough 0's to give the string length n. For example, the bad strings of length 5 are:

00000, 10000, 11000, 11100, 11110, 11111

In general, there will be $n+1$ *bad* strings of length n, and so, by the *Good* = *Total* - *Bad* rule, we get $g_n = 2^n - (n+1)$.

But, let's also find a recursive formula for g_n since that is what the problem asks for. To build a *good* string of length $n \geq 2$, we can begin with a 0 on the left end, and then write down any string of length $n-1$, except the all 0 string. So there are $2^{n-1} - 1$ ways to complete the string. On the other hand, if we begin with a 1 on the left, the following $n-1$ bits will have to be a *good* string of length $n-1$, so there are g_{n-1} choices for it. By the sum rule, we get $g_n = g_{n-1} + 2^{n-1} - 1$. For the initial condition, we have $g_1 = 0$.

Lesson 20: Selected Exercise Solutions

*1. Guess the solution to $a_0 = 0$, and $a_1 = 2$, and, for $n \geq 2$, $a_n = 4a_{n-1} - 3a_{n-2}$ and prove your guess is correct by induction.

Solution: Using the recursive formula, we see the sequence begins $0, 2, 8, 26, 80, 242$, and so it seems a likely guess is that $a_n = 3^n - 1$.

To prove that the guess is correct, let's use induction. For the basis case, $n = 0$, we have $a_0 = 0$ and $3^0 - 1 = 0$, so that checks out. We may as well check the case $n = 1$ separately as well since the recursive formula for a_n doesn't kick in until $n = 2$. So $a_1 = 2$ and $3^1 - 1 = 2$, so that does check as well.

Now for the inductive step, suppose $a_n = 3^n - 1$ for $n = 0, 1, \cdots j$ for some $j \geq 1$. Then

$$a_{j+1} = 4a_j - 3a_{j-1} = 4(3^j - 1) - 3(3^{j-1} - 1) = 4 \cdot 3^j - 4 - 3^j + 3 = 3 \cdot 3^j - 1 = 3^{j+1} - 1$$

as we needed to show. So the formula we guessed for a_n is correct for all $n \geq 0$.

*4. Find a closed form formula for the terms of the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

Solution: The characteristic equation for the recursion $f_n = f_{n-1} + f_{n-2}$ is $r^2 - r - 1 = 0$. Using the quadratic formula, the characteristic roots are found to be

$$r = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad r = \frac{1 - \sqrt{5}}{2}$$

So the general solution to the recursive formula is $f_n = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$.

To determine the values of α and β, we invoke the initial contions:

$$f_0 = 0 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^0 + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^0 = \alpha + \beta$$

and

$$f_1 = 1 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^1 + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^1 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right)$$

Multiplying the second equation by 2 we can rewrite is as $2 = \alpha(1 + sqrt5) + \beta(1 - \sqrt{5})$. So we need to solve the system

$$\alpha + \beta = 0$$
$$\alpha(1+\sqrt{5}) + \beta(1-\sqrt{5}) = 2$$

From the first equation, $\beta = -\alpha$. Putting that information into the second equation gives

$$\alpha(1+\sqrt{5}) - \alpha(1-\sqrt{5}) = 2$$

So

$$\alpha(2\sqrt{5}) = 2 \qquad \text{which means} \qquad \alpha = \frac{1}{\sqrt{5}}$$

.

So the closed form formula for the terms of the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

*6. Solve (use the general solution from part (a) when solving part (b).)

(a) $a_2 = 5, a_3 = 13$ and $a_n = 7a_{n-1} - 10a_{n-2}$, for $n \geq 4$.

(b) $a_2 = 5, a_3 = 13$ and $a_n = 7a_{n-1} - 10a_{n-2} + n$, for $n \geq 4$.

Solution (a): The characteristic eqaution is $r^2 - 7r + 10 = 0$. The left side factors as $(r-2)(r-5)$, and so the characteristic roots are $2, 5$. That means the general solution is $a_n = \alpha 2^n + \beta 5^n$. The initial conditions give the eqautions

$$5 = \alpha 2^2 + \beta 5^2 = 4\alpha + 25\beta$$
$$13 = \alpha 2^3 + \beta 5^3 = 8\alpha + 125\beta$$

Multiplying the first equation by 5, then subtracting the second gives $12 = 12\alpha$ so $\alpha = 1$, and that in turn means $\beta = \dfrac{1}{25}$.

So the solution is $a_n = 2^n + \frac{1}{25}5^n = 2^n + 5^{n-2}$ for $n \geq 2$.

Solution (b): To solve the nonhomogeneous recursion, we begin by determining a particular solution. Since the nonhomogeneous part of the recursive formula is n, we assume there is a particular solution of the form $a_n^{(p)} = An + B$. Putting that guess into the recursive formula we get

$$An + B = 7(A(n-1) + B) - 10(A(n-2) + B) + n$$

which means $An+B = 7An-7A+7B-10An+20A-10B+n$ or $(4A-1)n-13A+4B = 0$.

Since this must be true for any value of n we can conclude $4A-1=0$, so $A = \frac{1}{4}$, and $-13A+4B=0$ so that $B = \frac{13}{16}$.

Thus a particular solution is $a_n^{(p)} = \frac{n}{4} + \frac{13}{16}$.

Adding that to the general homogeneous solution found in part (a), we get the general solution to the nonhomogeneous recursion is $a_n = \alpha 2^n + \beta 5^2 + \frac{n}{4} + \frac{13}{16}$.

Using the initial conditions, we get two equations involving α and β.

$$5 = \alpha 2^2 + \beta 5^2 + \frac{2}{4} + \frac{13}{16} = 4\alpha + 25\beta + \frac{21}{16}$$

$$13 = \alpha 2^3 + \beta 5^3 + \frac{3}{4} + \frac{13}{16} = 8\alpha + 125\beta + \frac{25}{16}$$

Solving this system in the usual way, we get $\alpha = \frac{7}{12}$ and $\beta = \frac{13}{240}$.

So, the solution to the nonhomogeneous problem is

$$a_n = \frac{7}{12}2^n + \frac{13}{240}5^n + \frac{n}{4} + \frac{13}{16}$$

Whew!

*8. Solve (use the general solution from part (a) when solving part (b).)

(a) $a_0 = 2, a_1 = 5, a_2 = 15$, and $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, for $n \geq 3$.

(b) $a_0 = 2, a_1 = 5, a_2 = 15$, and $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} + 2n + 1$, for $n \geq 3$.

Solution (a): The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$, and, after a little experimentation we discover that $r = 1$ is a solution to that equation, and so $r-1$ must be a factor of the left side. (Do you recall the Factor Theorem from College Algebra? If not, look it up!) So we have $(r-1)(r^2-5r+6) = 0$, and it is easy to see how to complete the factoring: $(r-1)(r-2)(r-3) = 0$. So the characteristic roots are $r = 1, 2, 3$

That means the general solution is $a_n = \alpha 1^n + \beta 2^n + \gamma 3^n = \alpha + \beta 2^n + \gamma 3^n$.

The initial conditions require

$$2 = \alpha + \beta + \gamma$$

$$5 = \alpha + 2\beta + 3\gamma$$

$$15 = \alpha + 4\beta + 9\gamma$$

Solving in the usual way shows $\alpha = 1$, $\beta = -1$, and $\gamma = 2$, So the solution to the recurrence is $a_n = 1 - 2^n + 2(3^n)$

Solution (b): For the nonhomogeneous problem, let's again assume there is a particular solution of the form $a_n^{(p)} = An + B$ since that is the shape of the nonhomogeneous portion of the recurrence relation.

Putting that guess into the recurrence relation we get

$$An + B = 6(A(n-1) + B) - 11(A(n-2) + B) + 6(A(n-3) + B) + 2n + 1$$

when we simplify that equation it becomes $2A - 1 = n$, and there is no way this equation can be correct for all choices of n. So, following the suggestion in the text, we'll multiple our guess for a particular solution by n. So, now we assume $a_n^{(p)} = An^2 + Bn$.

Putting this new guess into the recurrence relation we get

$$An^2 + Bn = 6(A(n-1)^2 + B(n-1)) - 11(A(n-2)^2 + B(n-2)) + 6(A(n-3)^2 + B(n-3)) + 2n + 1$$

and this time things go better! In fact, we find $A = \frac{1}{2}$ and $B = \frac{9}{2}$ as you should check.

So now we can write down the general solution to the nonhomogeneous recursion as

$$a_n = \alpha + \beta 2^b + \gamma 3^n + \frac{1}{2}n^2 + \frac{9}{2}n$$

Invoking the initial conditions leads to the system

$$2 = \alpha + \beta + \gamma + \frac{1}{2}(0^2) + \frac{9}{2}(0)$$

$$5 = \alpha + 2\beta + 3\gamma + \frac{1}{2}(1^2) + \frac{9}{2}(1)$$

$$1 = \alpha + 4\beta + 9\gamma + \frac{1}{2}(2^2) + \frac{9}{2}(2)$$

Solving in the usual way we get $\alpha = 8$, $\beta = -10$, and $\gamma = 4$.

So the solution to the recurrence is $a_n = 8 - 10(2^n) + 4(3^n) + \left(\frac{1}{2}\right)n^2 + \left(\frac{9}{2}\right)n$.

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