UNIVERSITY OF NORTH DAKOTA
DEPARTMENT OF CORRESPONDENCE STUDY
DIVISION OF CONTINUING EDUCATION
Grand Forks, North Dakota

MATH 105
TRIGONOMETRY

Lessons:	14
Credit Hours:	Two(2)

TEXTBOOK (**required**):

Barnett, Ziegler, and Byleen *Analytic Trigonometry with Applications*
New York, New York: Wiley Publishing Company.
9^{th} Edition, 2006. ISBN 0-471-74655-X

Recommended: Student Solutions Manual. ISBN 0-471-74656-8

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(1/06)

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LESSONS:

INTRODUCTION

The material presented here constitutes a standard university level trigonometry course. In addition to the text, you might want to consider buying a copy of the *Student Solutions Manual* that accompanies the text. It contains detailed solutions to all the odd numbered problems in the text, as well as to all problems in the review sections at the end of each chapter. I think you will find it very helpful. **A scientific calculator is required for this course.** A suitable calculator will need to be able to compute the trigonometric functions (so it should have keys labeled sin, cos, and tan). Such calculators run about \$10-\$15. Graphing calculators are nice but tend to be pricy (\$75-\$100). **You do not need a graphing calculator for this course**, but you may use you one if you wish. There is no advantage to having a graphing calculator.

Your job for each lesson will be to read over the assigned sections in the text. Be sure you can follow all the details of each example presented there. As you read along, have a pencil and a piece of paper handy so you can fill in any missing steps in the examples. After reading the text, read through the Instructional Notes provided in this booklet. The notes repeat the important material of the sections covered in the lesson, usually with a slightly different point of view. The notes tend to be a little more informal than the text and a little sketchier, so it is best to read the text version first. Don't be in a hurry. Each of the 12 non-exam lessons covers about the amount of material that would be normal for one week in a regular class (in other words, about two hours of class time). So expect to spend at least two hours reading over the material for each lesson.

Before attempting the assignment to be submitted, I suggest you do the *warm-up* exercises listed at the end of each lesson in these notes. The warm-up exercises cover all the techniques you need to master for each lesson. The warm-ups will all be odd numbered problems so you can check your answers in the back of the text to make sure you are doing things correctly. If, after a lot of effort, you still cannot do a warm-up exercise, then it is time to turn to the Student's Solution Manual for a complete solution to the problem. Do not consult the Student's Solution Manual too quickly. You will <u>learn</u> only if you make every possible effort to do the problem on your own. Remember: You won't have a Student's

Solution Manual to guide you on the problems to be submitted nor on the examinations. So don't abuse it when doing the warm-up exercises. On the other hand, even if you do get the right answer on a warm-up problem, it might then still be a good idea to check out the solution in the Student's Solution Manual to see if a better method of solution is presented there.

After the warm-ups are finished tackle the **Problems to Submit**. Expect to spend two hours or so for these problems for each lesson. The assigned problems will all be even numbered ones, so you won't be able check your answers in the back of the book. Write the solutions up in a neat and logical fashion. An important part of a college math class is learning how to write in a mathematically mature way so take your time and think about what you are writing. You must show all the work leading to the answer so that the steps can be easily followed. A correct answer without correct reasoning will not get any credit.

The written assignments (warm-ups plus problems to be submitted) will probably provide enough practice solving problems to prepare you for the examination. However, if you feel you need more practice, pick out more of the odd numbered problems in the text to work on.

If you have access to a computer and the Internet, you might want to consider submitting your assignments directly through Blackboard as attachments. Details for attaching your homework problems are on the Lesson page in Blackboard. Assignments submitted via the Internet will normally be graded and returned within two to three days. Alternatively, you may mail paper versions of the written assignments to the Department of Correspondence & Online Study. They will be graded and returned to you. A turn-around time of 15 days or so seems to be typical for assignments sent via the post office. All grades will be posted in Blackboard once assignments are graded.

There will be 12 such written assignments. Each assignment will be graded on a scale from 0 to 100, and an overall average of the 12 assignments will be computed. That average will count as $\frac{1}{3}$ of your final grade.

If you have any questions about the reading, the examples, the homework problems or the instructional notes provided here, the fastest way to get help is via e-mail. You can send your questions to me at jerry.metzger@und.edu

I will normally answer e-mail within two days.

GRADING SYSTEM

There will be a total of 300 points distributed as follows:

Lesson 8: Examination I - Chapters 1,2,3 - 100 points

Lesson 14: Examination II - Chapters 4,5,6 - 100 points

Average of 12 written assignments: 100 points

Grades will be determined as follows:

270 - 300 total points - Final grade of A

240 - 269 total points - Final grade of B

210 - 239 total points - Final grade of C

180 - 209 total points - Final grade of D

0 - 179 total points - Final grade F.

LESSON 1

ANGLES AND SIMILAR TRIANGLES

READ:

The Introduction to these Notes

Chapter 1, Sections 1 and 2

INSTRUCTIONAL NOTES:

The familiar unit for measuring angles is the degree. There are 360 degrees in an angle corresponding to a complete rotation. The quantity 360 *degrees* is written as 360°. In half of a complete rotation there are 180°, and a quarter rotation has a measure of 90°.

Greek letters such as α (alpha), β (beta), γ (gamma), θ (theta), ϕ (phi), and ω (omega) are commonly used to denote angles. So, for example, if we wanted to say that α is an angle of 90 degrees, we would write $\alpha = 90^\circ$. Be sure to put the little degree symbol on the 90 to indicate the units being used to measure the angle. In latter sections we will be using other units to measure angles, so we need the degree symbol to keep things straight. Don't write $\alpha = 90$.

If an angle θ is drawn with its vertex at the center of a circle of radius r, then there is a simple relation between the quantities θ, r, and the length s of the circumference of the circle cut off by the sides of the angle. That relation is

$$\frac{\theta}{360^\circ} = \frac{s}{2\pi r}$$

where θ is measured in degrees. The quantity $2\pi r$ is the total circumference of a circle of radius r, so if we call that circumference C, the relationship can also be written as

$$\frac{\theta}{360^\circ} = \frac{s}{C}.$$

Using this equation, if we are given any two of the quantities θ, s, or C (or r in the first equation above), we can determine the remaining quantity.

A few fact from high school geometry will be needed from time to time. For example, in the footnote on page 7 of the text, you are reminded of the fact that when a line crosses two parallel lines, the corresponding angles are equal. In section 1.2 of the text, the theorem that the ratio of corresponding lengths in similar triangles are equal (Euclid's Theorem) is put to use to compute distances that would be difficult to measure directly.

The homework problems come in two types. The **Warm-up** exercises will all be odd numbered problems. The answers to these can be checked in the back of the text, and more detailed solutions can be found in the *Student's Solution Manual.* These warm-up problems should provide sufficient practice in the techniques of each lesson so you can do the **Problems to Submit** assignment. Do not send in the warm-up problems for grading. If you find some of the suggested warm-up problems difficult, select a few more odd numbered ones of the same type to work out. The problem sets are divided into groups of similar types of problems, so picking a few more from the same group is all you need to do.

There will be 10 (more or less) problems to submit for each lesson. Write up the solutions in a neat well organized fashion showing the steps that lead to your answer. The homework grades will form a significant portion of your course grade, so take your time and do a good job on these assignments. Also, the questions on the two examinations will be based on the material covered in the homework, so these assignments will help you know what to expect on those tests.

Suggested Warm-up Assignment:

Section 1.1: 1,3,17,23,41,45, 47.

Section 1.2: 5,11,13,23.

Problems to Submit:

Section 1.1: 2,4,18,24,42,46,48.

Section 1.2: 6,12,14,24.

LESSON 2

TRIGONOMETRIC RATIOS AND RIGHT TRIANGLES

READ:

Chapter 1, Sections 3 and 4

INSTRUCTIONAL NOTES:

In a triangle there are three sides and three angles. One of the basic problems of trigonometry is to determine some of these quantities from given values for the others. We will have to be given enough information to determine the triangle of course. For example, if we are told the three angles, we would be stuck if we wanted to compute the lengths of the sides since a triangle can be shrunk or magnified without changing the angles. By the way, note that if we know two angles in a triangle, we can compute the third angle because the sum of the three angles is 180°. On the other hand, if we are told the two legs of a right triangle, we can use the Theorem of Pythagoras to calculate the length of the hypotenuse, and, using methods we will look at in this section, we will be able to find the other two angles as well. The process of determining the unknown measurements in a triangle from the known ones is called *solving the triangle.* In this lesson, we will concentrate on solving only right triangles.

In the last lesson, we used ratios of corresponding sides of similar triangles to solve some triangles. Now we will concentrate on right triangles and use a calculator to determine the ratios for us.

The method of solution for right triangles makes use of the ratios of the sides of a right triangle with an acute angle we will name θ. (See the diagram in the text, bottom page 23.) There are six ratios named sine, cosine, tangent, cotangent, secant, and cosecant. These six quantities tell us what the ratios will be for various pairs of sides of a right triangle with a given acute angle θ. You need to memorize the formulas for these six ratios as given in the table on page 23.

As a first example, imagine a right triangle with the two legs each having length 1 unit.

Since the legs have the same length, the angles opposite those legs must also be equal, and, since they have to add up to $180^\circ - 90^\circ = 90^\circ$, each must be 45° angles. The length of the hypotenuse is $\sqrt{1^2+1^2} = \sqrt{2}$. Now concentrate on one of those 45°. Draw a diagram! Notice that the side adjacent to that angle has length 1 and the hypotenuse has length $\sqrt{2}$. So, according to the definition of the sine given in the table, we see $\sin 45^\circ = \frac{1}{\sqrt{2}}$. Going through all six ratios we get

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = \frac{1}{1} = 1$$

$$\sec 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \csc 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \cot 45^\circ = \frac{1}{1} = 1$$

Don't let the orientation of the triangle confuse the computations. The text drew the triangle with the angle θ at the lower left of the picture. But it could just as well have denoted the other acute angle. The important feature of the definition of the six trigonometric ratios is the relationship between the angle and the sides. For example, the sine of an acute angle in a right triangle is always the ratio of the length of the side opposite that angle to the length of the hypotenuse.

Normally, it is best to use exact values for the trigonometric ratios whenever possible. So, for example, in the table above, we see $\sec 45^\circ = \sqrt{2}$. That is usually better that writing $\sec 45^\circ = 1.414$, or some other decimal approximation. In fact let's agree to **never use decimal approximations when exact values are possible**.

Be sure to include the name of the angle when you refer to a trigonometric ratio. In the table above, don't write tan = 1. That should prompt the question, *the tangent of what angle is equal to 1?* Likewise, when dealing with a general angle, say θ, with a sine equal to .3 say, don't write sin = .3! Correct is $\sin\theta = .3$.

Armed with a calculator that can compute the trigonometric ratios for various angles, we can solve right triangles without much effort. Practice the examples in section 1.3. More realistic problems are presented in section 1.4, where again right triangles are solved, but now the triangles are described verbally, so we need to first construct a diagram from the description and then solve the right triangle that appears. Such problems are one step more complex than the problems of section 1.3. Be sure you draw a diagram for such problems.

Suggested Warm-up Assignment:

Section 1.3: 13,17,29,31,43,53.

Section 1.4: 5,19,31.

Problems to Submit:

Section 1.3: 14,22,30,32,44,54.

Section 1.4: 6,20,32.

LESSON 3

RADIAN MEASURE AND TRIGONOMETRIC FUNCTIONS

READ:

Chapter 2, Sections 1 and 3 (we will omit Section 2.2)

INSTRUCTIONAL NOTES:

Besides degrees, there is another measure of angle commonly used in trigonometry. The *radian* measure could be defined by saying an angle of $360°$ is an angle of 2π radians. In other words, if we rotate one complete turn, we have moved through an angle of 2π radians; that is, just a hair more than 6.28 radians. Half a turn would be π radians.

When it is clear we are talking about angles, the word *radians* is dropped, so we would just say a half turn is an angle of π. A right angle, in other words a quarter turn, would be an angle of $\frac{\pi}{2}$. Note that 1 radian is equal to $\frac{360}{2\pi}$ degrees, which is roughly $57°$.

The formula connecting arc length, s, radius, r, and central angle, θ in a circle is

$$\frac{\theta°}{360°} = \frac{s}{2\pi r}.$$

If we express this same equation in radian measure, it looks like $\frac{\theta}{2\pi} = \frac{s}{2\pi r}$, or, canceling the 2π's, $\theta = \frac{s}{r}$. That equation can also be written as

$$s = r\theta \qquad \theta \text{ measured in radians, an equation worth remembering!}$$

It is convenient to place angles in a particular *standard position* in the x, y-coordinate plane. The standard position has the vertex of the angle at the origin and the initial side of the angle along the positive x-axis. The terminal side of the angle is represented by a ray emanating from the origin, and we think of the angle as being formed as we rotate the ray representing the terminal side starting from the initial side of the angle to its final position. See the diagrams in the text, page 58. We will define a positive angle as one produced by a counterclockwise rotation and a negative angle as one produced by a clockwise rotation. For example, an angle of $90°$ or $\frac{\pi}{2}$ would have its terminal side along the positive y-axis while an angle of $-90°$ or $-\frac{\pi}{2}$ radians would have its terminal side along the negative y-axis.

We can think about angles bigger than 360° degrees. For example, if we make one and a half complete turns in the positive direction, the terminal side of that angle would be along the negative x-axis. That would be an angle of $360° + 180° = 540°$ or $2\pi + \pi = 3\pi$ radians.

In which quadrant would the terminal side of an angle of 78 radians lie? Solution: Since $\frac{78}{2\pi} = 12.41\cdots$, we see that 78 radians make 12 complete turns and $.41\cdots$ of a 13^{th} turn. Since .41 is more than a quarter but less than a half, we conclude the terminal side is in the second quadrant, Q_2.

Let A be the area of a circular sector with a central angle of θ radians in a circle of radius r. See the diagram in the text, page 60. Since the total area of the circle is πr^2 and since we are only going the proportion $\frac{\theta}{2\pi}$ of the way around the circle, we can conclude that $A = \frac{\theta}{2\pi}\left(\pi r^2\right)$. In other words, after canceling, we see,

$$A = \frac{1}{2}r^2\theta, \qquad \theta \text{ measured in radians, another equation worth remembering!.}$$

Notice that both this equation for area and the earlier one for arc length are wrong if θ is measured in degrees.

From now on, it will be important to think about whether an angle is being given in degrees or radians. Be particularly careful that your calculator is set in the correct *mode*, either degree or radian as appropriate, when using the sin, cos, or tan keys. Otherwise, completely bogus answers will result from computations.

In lesson 1, we defined the six trigonometric ratios for an acute angle in a right triangle. It turns out to be useful to define the sine, cosine, and the other trigonometric functions for any angle, not just acute angles. These extensions of the definitions of the trigonometric functions is carried out using the *unit circle*: the circle of radius 1 with center at $(0, 0)$ in the coordinate plane. Since we will be using x to denote the angle, we can't use it for the name of the horizontal axis. We will use a and b to designate the horizontal and vertical axes respectively. So the unit circle has equation $a^2 + b^2 = 1$.

Consider any angle x drawn in standard position as in the diagram on page 75 of the text. Let Q be a point on that terminal side (different from the origin) with coordinates (a, b). We *define* the six trigonometric functions of x as follows:

$$\sin x = b \qquad \cos x = a$$

$$\tan x = \frac{b}{a} \quad \text{provided } a \neq 0 \qquad \sec x = \frac{1}{a} \quad \text{provided } a \neq 0$$

$$\csc x = \frac{1}{b} \quad \text{provided } b \neq 0 \qquad \cot x = \frac{a}{b} \quad \text{provided } b \neq 0$$

In the last four ratios, we assume a or b is not 0 as needed. For example, if $a = 0$, then the trigonometric ratios tangent and secant are **undefined** for the particular angle x. Note that $a = 0$ will happen when the terminal side the angle is along either the positive or negative b-axis.

Since the first and second coordinate of Q is between -1 and 1, we see the value of $\sin x$ must always be between -1 and 1. Likewise, the values of $\cos x$ must be between -1 and 1.

Example 1, page 76 of the text shows how the steps outlined above would go for a particular point P . Study that example carefully! Pay special attention to the way in which the point Q on the unit circle is determined. That is always the first step when determining the values of the trigonometric functions.

As one last example, let's compute the six trigonometric functions of the number $\frac{\pi}{2}$. In this case, instead of being handed a point on the terminal side of a angle, we are just told the angle, and it's up to us to find a point on the terminal side. The terminal side of an angle of $\frac{\pi}{2}$ radians will be along the positive b-axis. Let's use the point $P = (0, 1)$ on the terminal side. This point is already on the unit circle, so we can *cut to the chase.*

$$\sin\frac{\pi}{2} = \frac{1}{1} = 1 \qquad \cos\frac{\pi}{2} = \frac{0}{1} = 0 \tan\frac{\pi}{2} \qquad \text{is undefined}$$

$$\sec\frac{\pi}{2} \qquad \text{is undefined} \csc\frac{\pi}{2} = \frac{1}{1} = 1 \qquad \cot\frac{\pi}{2} = \frac{1}{1} = 1$$

Notice that $\tan\frac{\pi}{2}$ and $\sec\frac{\pi}{2}$ are <u>undefined</u> since each involves a quotient with a zero denominator, and division by zero is an undefined operation.

Suggested Warm-up Assignment:

Section 2.1: 7,19,31A,43,55.

Section 2.3: 7, 13,17, 23,25,27,43

Problems to Submit:

Section 2.1: 8,20,32A,46,56.

Section 2.3: 8,14,18,24,26,28,44.

LESSON 4

REFERENCE ANGLES AND SPECIAL ANGLES

READ:

Chapter 2, Section 5, to page 106

INSTRUCTIONAL NOTES:

For most values of x, you will need a calculator (or, in the olden days, a table in the back of the book) to compute the value of a trigonometric function of x. There is no convenient way to compute something like $\tan .3$ by hand. But a calculator gives $\tan .3 = 1.26\cdots$ in a flash. (At least if you remember to put the calculator in radian mode, and are willing to accept a decimal approximation for the correct answer!)

However, there are certain values of x for which it is possible to easily compute the exact values of the six trigonometric functions. These numbers occur so often that you will need to memorize them. Actually that is not as bad as it might seem since, as you will see, there is a lot of symmetry to the list of numbers you need to learn.

In all that follows, we will use radians, but everything works just as well for degrees. The easiest x's to handle are the four *cardinal* or *quadrantal* values 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. If you draw the x,y-coordinate system and think of these numbers as angles in radians in standard position, you'll see they correspond to the compass direction of east, north, west, and south respectively.

In lesson 3, we found $\sin\frac{\pi}{2} = 1$. We did that by selecting a point on the terminal side of the angle, at one unit distance from the origin. The remaining five trigonometric function of $\frac{\pi}{2}$ are all easy to compute in the same way. (You should do that now!) And the process can be repeated for the other three cardinal values. (You should do that now, too!) The results are given here (*und.* means undefined):

$$\sin 0 = 0 \quad \cos 0 = 1 \quad \tan 0 = 0 \quad \csc 0 \ \textit{und.} \quad \sec 0 = 1 \quad \cot 0 \ \textit{und.}$$

$$\sin\frac{\pi}{2} = 1 \quad \cos\frac{\pi}{2} = 0 \quad \tan\frac{\pi}{2} \ \textit{und.} \quad \csc\frac{\pi}{2} = 1 \quad \sec\frac{\pi}{2} \ \textit{und} \quad \cot\frac{\pi}{2} = 0$$

$$\sin\pi = 0 \quad \cos\pi = -1 \quad \tan\pi = 0 \quad \csc\pi\ \textit{und.} \quad \sec\pi = -1 \quad \cot\pi\ \textit{und.}$$

$$\sin\frac{3\pi}{2} = -1 \quad \cos\frac{3\pi}{2} = 0 \quad \tan\frac{3\pi}{2}\ \textit{und.} \quad \csc\frac{3\pi}{2} = -1 \quad \sec\frac{3\pi}{2}\ \textit{und} \quad \cot\frac{\pi}{2} = 0$$

The information above can be used to compute the trigonometric functions of other cardinal values. For example, 3π, as an angle in radians in standard position, has terminal side along the negative x-axis just as π does. And so $\sin 3\pi = \sin\pi = 0$. Likewise, $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ and $\frac{\pi}{2}$ have the same terminal side, and so we may conclude that, for example, $\sec\frac{5\pi}{2} = \sec\frac{\pi}{2}$ is undefined. In general, any trigonometric function of any value of x that gives a terminal side along one of the coordinate axes can be computed from the table above.

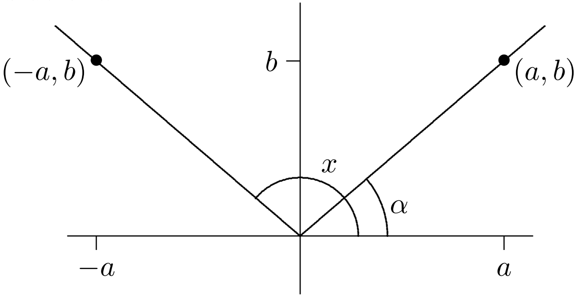
There are three other basic values for which computing the trigonometric functions is easy. They are based on two familiar right triangles, as drawn in figure 6, page 103 of the text. One is a right triangle in which both legs have length 1, so that the hypotenuse has length $\sqrt{2}$. Since the two acute angles in this triangle are opposite sides of equal length, the two angles must be equal. That means each must have radian measure $\frac{\pi}{4}$. To form the other right triangle, start with an equilateral triangle with sides all of length 2. Since the sides all have the same length, the angles must all be equal, and so they are each of radian measure $\frac{\pi}{3}$. Now form a right triangle by dropping a line from the top vertex perpendicular to the opposite side. The result is a right triangle with acute angles of $\frac{\pi}{3}$, from the original triangle, and $\frac{\pi}{6}$, from bisecting the top angle in the original triangle.

The is a handy table showing the values of the trigonometric functions at the six *special angles* on the lower left corner of the page inside the front cover of the text. By looking at the two triangles described above, the middle three rows of that table should be pretty clear. Notice that according to the definitions of the trigonometric functions, $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\cot x = \frac{1}{\tan x}$. That means it is only necessary to remember the three columns labeled $\sin x$, $\cos x$, and $\tan x$, since the other three columns can be produced by taking reciprocals of the values in these three columns. Actually, noticing that $\tan x = \frac{\sin x}{\cos x}$, the entries in the tangent column can be computed by a quick division if we know the values in the sine and cosine columns.

There is a neat pattern to the entries in the sine column which can help get it right. The entries in order are $0 = \frac{\sqrt{0}}{2}$, $\frac{1}{2} = \frac{\sqrt{1}}{2}$, $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, and $1 = \frac{\sqrt{4}}{2}$. So the denominators are all 2 and the square roots in the numerator are of $0, 1, 2, 3$, and 4. Finally, notice that the cosine column is just the same as the sine column, except it is upside down. Well, that's a few different ways of looking at the numbers in the table. **What ever way you select, the table needs to be memorized**.

Knowing the values of the trigonometric functions as given in the table permits the exact evaluation of the trigonometric functions for many values of x related to those basic values. To see how this works, we begin with a definition: For a value x (not equal to a cardinal value) the *reference angle* is the acute angle between the terminal side of an angle of x radians in standard position and the horizontal axis. In the diagrams on page 105 in the text, the reference angle in each case is labeled α.

Now suppose we are given a value x with reference angle α. We can compute the six trigonometric functions of x if we know the six trigonometric functions of α. Here's how to do that. Draw the angle α in standard position. Pick a point $(a, b) \neq (0, 0)$ on the terminal side of α. Using the symmetry of the diagram, we can see that a point on the terminal side of x will have coordinates related to (a, b). For the one in the diagram below, we see $(-a, b)$ will be on the terminal side of x.



In this example,using $R = \sqrt{a^2 + b^2}$, we see

$$\sin x = \frac{b}{R} \qquad \cos x = -\frac{a}{R} \qquad \tan x = -\frac{b}{a}$$
$$\csc x = \frac{R}{b} \qquad \sec x = -\frac{R}{a} \qquad \cot x = -\frac{a}{b}$$

On the other hand

$$\sin \alpha = \frac{b}{R} \qquad \cos \alpha = \frac{a}{R} \qquad \tan \alpha = \frac{b}{a}$$
$$\csc \alpha = \frac{R}{b} \qquad \sec \alpha = \frac{R}{a} \qquad \cot \alpha = \frac{a}{b}$$

The important thing to notice about this chart is that the trigonometric functions of x and of the reference angle are always the same, except possibly for the sign. Since the reference angle is in the first quadrant, the values of the trigonometric functions of the reference angles will always be positive. But for x itself, the functions might have either positive or negative values. The story is the same if x happens to have its terminal side in any of the four quadrants.

All this means that to compute the value of a trigonometric function of a number x, we can just compute the value of the trigonometric function of the reference angle, and then attach the correct sign (either $+$ or $-$). For example, if we wanted to compute $\sin \frac{7\pi}{4}$, we note the reference angle is $\frac{\pi}{4}$. Now $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. So we know that either $\sin \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$ or $\sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$. Recalling that the sine has negative values in the fourth quadrant (see page 75 of the text), we can conclude $\sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$.

One more example: Let's compute $\tan\left(-\frac{10\pi}{3}\right)$. Now $-\frac{10\pi}{3}$ is a second quadrant angle with reference angle $\frac{\pi}{3}$ (draw a picture to see why that is correct), and $\tan \frac{\pi}{3} = \sqrt{3}$. Also, the tangent is negative in the second quadrant. Thus we can conclude $\tan\left(-\frac{10\pi}{3}\right) = -\sqrt{3}$.

In general, if x is in the first quadrant, and (a, b) is a point on the terminal side of the reference angle for x, then (a, b) will be on the terminal side of x. If x is in the second quadrant, the point will be $(-a, b)$, for the third quadrant, it will be $(-a, -b)$. And for the fourth quadrant, it will be $(a, -b)$. After you have worked through a few dozen such examples, the steps (1. find reference angle, 2. determine trigonometric function of the reference angle, 3. determine the correct sign) can be done mentally pretty quickly. As we go on in the text, we will learn many other facts about the trigonometric functions which make these computations even easier.

We will also want to be able to solve the following sort of problem: Suppose that $\cos x = \frac{1}{2}$, and that x is between 0 and 2π. What are the possible values of x? This might be termed the *inverse problem* to the one described in the previous paragraphs. This time we are told the value of the trigonometric function and want to determine the x. Thinking about the table learned earlier, we see we need values of x for which the reference angle is $\frac{\pi}{3}$. Hence $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$. Since the cosine is positive in the first and fourth quadrants, we conclude the values of x, subject to the restriction $0 \le x < 2\pi$, will be $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Be sure to give exact values, not decimal approximations, when possible!

Suggested Warm-up Assignment:

Section 2.5: All odd numbered problems 13-41, 73,81.

Problems to Submit:

Section 2.5: All even numbered problems 14-42, 52,74,82.

LESSON 5

PERIODICITY AND INDENTITIES

READ:

Chapter 2, Section 5, page 106 to the end

INSTRUCTIONAL NOTES:

There are a few important fact to notice about the new way of thinking about the trigonometric functions introduced in Lesson 3:

1. By computing the coordinates of the stopping point when we measure off a length x on the circumference of the unit circle, we immediately have the values of $\cos x$ and $\sin x$. The other four trigonometric functions of x are computed from these two values.

2. Since the coordinates of the points on the unit circle vary between -1 and 1, the values of the sine and cosine functions also vary between -1 and 1. For example, it isn't possible to find an x for which $\sin x = 2$.

3. Think about following a point as it moves around the unit circle counterclockwise starting at the point $(1, 0)$. As noted above, $y = \cos x$ keeps track of the first coordinate of that point. The y value will start at 1 when $x = 0$ and begin to decrease, eventually reaching $y = 0$ when the point reaches the vertical axis. As we continue to follow the point, the first coordinate continues to decrease, finally reaching a value of $y = -1$ when the point gets to the negative side of the horizontal axis. Draw a diagram and convince yourself that this description is correct! As we continue following the moving point, the first coordinate (that is, the cosine) starts increasing. When the point has gone all the way around to the positive horizontal axis, the value of $y = \cos x$ is back up to 1 again when $x = 2\pi$. If we allow the point to continue rotating around the circle, the quantity $y = \cos x$ will cycle through the same values again. Briefly, we say the cosine function is 2π-*periodic.* Of course, the values of the sine function exhibit the same cycling pattern, except $y = \sin x$ starts out at $y = 0$ when the point we are following begins on the positive horizontal axis.

4. Since the total distance around the unit circle is 2π, we see that for any number x,

$\cos x = cos(x + 2\pi)$, and $\sin x = \sin(x + 2\pi)$ since marking off an arc length of x on the unit circle leaves us at exactly the same spot as marking off an arc length of $x+2\pi$. In fact, going around an extra 2π or 4π, or 6π, and so on, will always leave us at the same stopping point on the unit circle, and consequently that same values for the sine and cosine functions. In other words, for <u>any integer k</u>, $\sin x = \sin(x + 2k\pi)$ and $\cos x = \cos(x + 2k\pi)$. To repeat: because the sine and cosine functions cycle through the same values at an interval of 2π, we will say these two functions are 2π-periodic.

5. Since, for any number x, the point $(\cos x, \sin x)$ is located on the unit circle, it satisfies the equation for that circle. In other words we see that, for any number x, $\cos^2 x+\sin^2 x = 1$.

6. On the unit circle, mark off an angle of x radians and one of $-x$ radians, each in standard position. If you look at the coordinates of the two ending points, you will see that if the coordinates of the first are (a, b), then the coordinates of the second are $(a, -b)$. The first coordinate stays the same and the second coordinate changes sign. Thinking about the definitions of the trigonometric functions, we see that means

$$\cos(-x) = \cos x \qquad \text{and } \sin(-x) = -\sin x.$$

Since $\tan x = \frac{\sin x}{\cos x}$, we also see that $\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$. The various relations listed above can be used to simplify expressions involving the trigonometric functions. Here are a few examples to show the method:

$$\cos x \sec x = \cos x \left(\frac{1}{\cos x}\right) = 1.$$

$$\frac{\cos x}{1-\sin^2 x} = \frac{\cos x}{(\sin^2 x + \cos^2 x) - \sin^2 x} = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x.$$

$$\tan(-x)\cos(-x) = (-\tan x)(\cos x) = -\frac{\sin x}{\cos x}\cos x = -\sin x.$$

Suggested Warm-up Assignment:

Section 2.5: 57, 59, 5, 67, 69, 71, 85, 87

Problems to Submit:

Section 2.5: 58,60,62,64,66,68,70,72,86,88.

LESSON 6

GRAPHING THE TRIGONOMETRIC FUNCTIONS

READ:

Chapter 3, Sections 1 and 2

INSTRUCTIONAL NOTES:

The functions $y = \sin x$, $y = \cos x$, $y = \sec x$, and $y = \csc x$ are each 2π-periodic. That means when the graph of one of these functions is drawn for x values in an interval of length 2π, that graph will repeat the same pattern in the next interval of length 2π, and again in the next interval of length 2π, and so on. For that reason, it is only necessary to draw the graphs of these four functions for x's in one interval of length 2π. The interval we will use is $[0, 2\pi]$. The functions $y = \tan x$ and $y = \cot x$ are π-periodic, so these two need only be graphed for x's in an interval of length π to show how the entire graph will look.

In algebra classes, we learned to draw graphs by plotting points. For example, to draw the graph of $y = x^2 + 1$, we would note that when $x = 0$, $y = 1$, so that $(0, 1)$ will be one point on the graph. Likewise $(-1, 2)$, $(1, 2)$, $(-2, 5)$, and $(2, 5)$ will be more points on the graph. After plotting many such points, we play *connect the dots* drawing a smooth curve through the points. In this case, the graph is a parabola opening upwards, with its vertex on the y axis one unit above the origin.

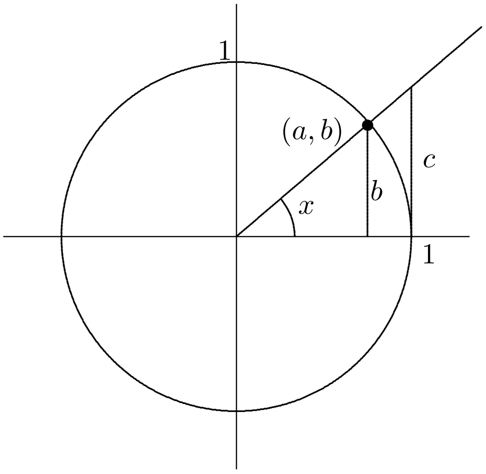
We could go about drawing the graph of a trigonometric function such as $y = \sin x$ in exactly the same way. In fact, if you have a graphing calculator, that is exactly what it would do. It would plot maybe several hundred such points, the specific number depending on the resolution of the display. When all the dots are in place, the result will look like a solid curve. While it is possible for us to mimic the calculator's approach by plotting many points, it is much better for us to use brain power, an attribute the calculator lacks.

For example, to draw the graph of $y = \cos x$, we recall that the cosine function is simply the first coordinate of a moving point on the unit circle as we follow it in the counterclockwise direction starting at the point $(1, 0)$. With that in mind, it should be almost instantly clear that the graph of $y = \cos x$ looks like the picture on page 128 of the text. Similarly, recalling the $y = \sin x$ keeps track of the second coordinate, we see the graph of that function will look like the diagram on page 127 of the text.

To find the graph of $y = \sec x$, we will use the relation $\sec x = \frac{1}{\cos x}$. That means that for each x, the y value on the secant graph will be the reciprocal of the corresponding value on the cosine graph. Where the cosine has value 1, the secant has value $\frac{1}{1} = 1$; where the cosine has value $\frac{1}{2}$, the secant will have value $\frac{1}{\left(\frac{1}{2}\right)} = 2$; where the cosine has value $-\frac{2}{3}$, the secant will have value $-\frac{1}{\left(\frac{2}{3}\right)} = -\frac{3}{2}$, and so on. Notice that when the cosine has a value of 0, the secant is undefined since we cannot form the reciprocal of 0. When the values of the cosine are very close to 0, but just a hair positive, the values of the secant will be huge and positive. For example, at a place where the value of the cosine is .001, the secant will have a value of 1000. Putting all these thoughts together, you should see precisely why the graph of $y = \sec x$ looks like the bottom graph on page 133 of the text.

In the same fashion, noting that $\csc x = \frac{1}{\sin x}$, the shape of the graph of $y = \csc x$ given in the top diagram on page 133 of the text should also make complete sense.

The text explains one way to determine the shape of the graph of $y = \tan x$. Here is another way that may be a little easier to think about. Consider the diagram below. The vertical line through $(1, 0)$ is tangent to the circle, and c denotes the length of the line from $(1, 0)$ up to the point where is meets the ray extending from the origin.



By looking at similar triangles, we see that $\frac{c}{1} = \frac{b}{a}$. But $\frac{b}{a} = \tan x$. Thus, $c = \tan x$. So, when we draw the graph of $y = \tan x$, we are just recording the length c for various values of x, at least when x is between 0 and $\frac{\pi}{2}$. We see the line starts with a length of 0 when $x = 0$, and gets longer and longer as x gets closer and closer to $\frac{\pi}{2}$. For x between $-\frac{\pi}{2}$ and 0, the story is pretty much the same, except $\tan x$ will be $-c$ which equals the negative of the length of the corresponding line in quadrant four. Draw a picture, and convince yourself of the correctness of that statement. Putting all these thoughts together, the shape of $y = \tan x$ as given on page 131 of the text should make sense.

Finally, using $\cot x = \frac{1}{\tan x}$, the graph of $y = \cot x$ can be produced by taking the reciprocals of the corresponding values on the tangent graph as we did above for the secant and cosecant. The finished product appears on page 132 of the text.

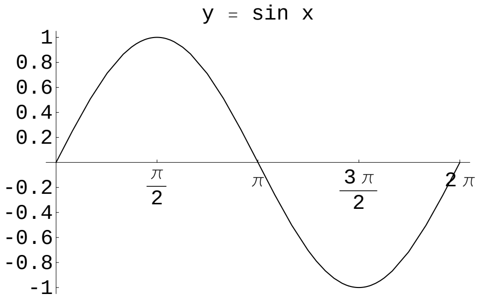
You need to be able to reason out what the graphs of the six trigonometric functions look like reasonably quickly. You should also be able to determine the various interesting features of each graph. The interesting features include the x-intercepts. For example, the x-intercepts for $y = \sin x$ occur at the integer multiples of π: $\cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$. The sine and cosine functions have a maximum value of 1 and a minimum value of -1. You need to be able to determine the x values where these maxima and minima occur.

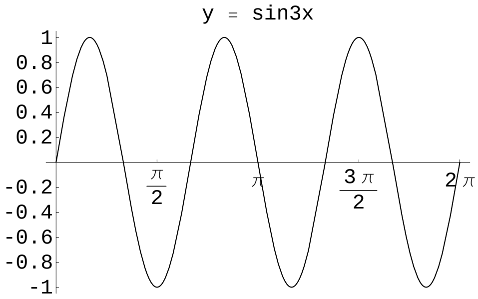
Graphing a more complicated equation involving trigonometric functions such as $y = 5\sin 3x$ could be done by the *plot-lots-of-points-and-connect-the-dots* method. But it is much better to see how such a function can be built up in easy stages from the basic graphs we just learned practiced. Let's break down the function above into bite-sized pieces.

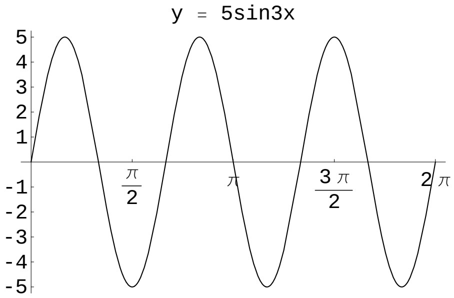
First note that the graph of $y = \sin 3x$ will look like the graph of $y = \sin x$, except it will go through a period three times as fast. In other words, it will begin a cycle when $3x = 0$, that is, when $x = 0$, and it will complete that cycle when $3x = 2\pi$, that is, when $x = \frac{2\pi}{3}$. That means the new function goes through its cycles 3 times as fast as the original, starting a cycle at $x = 0$ and completing it at $x = \frac{2\pi}{3}$. The period of $y = \sin 3x$ is $\frac{2\pi}{3}$.

Now the graph of $y = 5\sin 3x$ can be obtained from the graph of $y = \sin 3x$ by noting that the y values on the new graph are just 5 times the values of y on the old graph. In other words, the coefficient of 5 stretches the first graph vertically, so that instead of seeing values between -1 and 1, we will be getting values between -5 and 5. The stretching is expressed by saying the *amplitude* for the new graph is 5.

Here are the three graphs:







Suggested Warm-up Assignment:

Section 3.1: 3,5,7,15,17,19.

Section 3.2: 3, 9,11,13,23,27.

Problems to Submit:

Section 3.1: 4,6,8,16,18,20.

Section 3.2: 4,10,12,14,24,28.

LESSON 7

GRAPHING FANCIER TRIGONOMETRIC FUNCTIONS

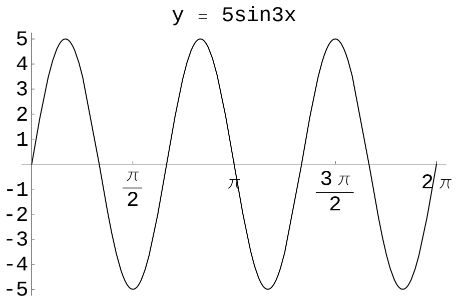
READ:

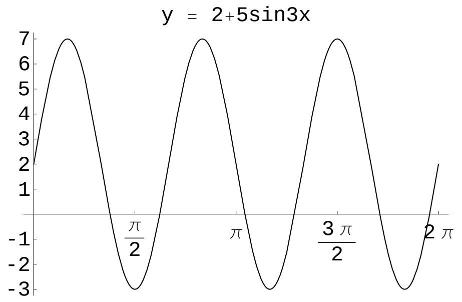
Chapter 3, Sections 3 and 6 (omit sections 4 and 5)

INSTRUCTIONAL NOTES:

Graphing even more complicated equation involving trigonometric functions such as $y = 2 + 5\sin 3x$ builds on the method practiced in the last section.

We already know what the graph of $y = 5\sin 3x$ looks like. The graph of $y = 2+5\sin 3x$ can be obtained from the graph of $y = 5\sin 3x$ by noting that the y values on the new graph are produced by adding 2 to the y values on the old graph. In other words, the effect of the 2 is to lift the original graph up 2 units. Here are the two graphs side by side.





In general, to draw the graph of $y = k + A\sin Bx$ or $y = k + A\cos Bx$, consider the effect on the graph of each parameter in the order B, (which changes the period), A, (which changes the amplitude), and then k, (which moves the graph up or down). Stick with the indicated order since other orders can be trickier to analyze.

Only slightly more complicated is graphing functions such as $y = 2 + 5\sin(3x - 4)$ or $y = 2 + 5\cos(3x - 4)$. The only difference comes in the first step. When determining the new period, we know a period will start when $3x - 4 = 0$, that is, when $x = \frac{4}{3}$, and the cycle ends when $3x - 4 = 2\pi$, that is, when $x = \frac{4}{3} + \frac{2\pi}{3}$. All that means is that one cycle begins at $\frac{4}{3}$ and ends $\frac{2\pi}{3}$ later. So the length of the period is still $\frac{2\pi}{3}$ as in the last example, but now the cycle begins at $x = \frac{4}{3}$ rather than at $x = 0$. The net effect is that the graph of $y = 2 + 5\sin(3x - 4)$ looks just like the graph of $y = 2 + 5\sin 3x$ we constructed above, except the entire picture has been pushed to the right $\frac{4}{3}^{rd}$ of a unit. The number $\frac{4}{3}$ is called the *phase shift* for the graph.

Graphs of functions of the forms $y = A\tan(Bx + C)$ and $y = A\cot(Bx + C)$, are constructed using the same ideas employed for graphs of functions like $y = A\sin(Bx + C)$. There are a few small differences you need to keep in mind however. First, these two new functions do not have amplitudes. The effect of the coefficient A is still to stretch the values of the graph vertically, but the amplitude (which equals one half of the difference between the maximum value of the function and the minimum value of the function) has no meaning for these functions which have no maximum or minimum. Second, the tangent and cotangent functions are π-periodic instead of 2π-periodic. That means that if we wanted to graph $y = 2\tan(3x + \frac{\pi}{4})$, we would find the start and end of one period by solving $3x + \frac{\pi}{4} = 0$ and $3x + \frac{\pi}{4} = \pi$. Note: for a cotangent graph, it is most convenient to use $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for the values on the right sides of those two equations since between those two values the cotangent's graph comes in one piece.

Functions such as $y = A\sec(Bx + C)$ and $y = A\csc(Bx + C)$ could be graphed using the methods described above, but an indirect route is usually easier. Look back in the last lesson to see how the graphs of $y = \sec x$ and $y = \csc x$ were constructed from the graphs of $y = \cos x$ and $y = \sin x$ *by taking reciprocals.* The same technique can be used to graph a function like $y = 2\sec(x + \pi)$. The idea is to first graph $y = \left(\frac{1}{2}\right)\cos(x + \pi)$, and then take reciprocals to produce the secant graph we really want. Note the coefficient 2 be comes $\frac{1}{2}$ in the cosine function.

Suggested Warm-up Assignment:

Section 3.3: 11,15,25,27.

Section 3.6: 7,9,11,21.

Problems to Submit:

Section 3.3: 12,16,26,28.

Section 3.6: 8,10,12,22.

The next lesson is the midterm examination. You may submit the *Request for Examination* form with this assignment if you wish. See the next lesson for some information about the test.

LESSON 8

MIDTERM EXAMINATION

ABOUT THE TEST:

The examination will cover Chapter 1, Sections 1 trough 4, Chapter 2, Sections 1, 3, 5, and Sections 1, 2, 3, and 6 of Chapter 3. There will be 20 questions on the test. You will have $2\frac{1}{2}$ hours for the examination, but it probably won't take you that long if you are well prepared. The problems will all be very much like the homework problems you have been submitting. There will be no surprises!

To prepare for the examination, you should go over the homework that has been graded and returned to you. Pay particular attention to corrections that appear on your homework. If you want more practice, each chapter in the text is followed by a section of *Review Problems* that might be worth looking over. The review section problems have complete solutions written up in the Student Solutions Manual.

No books, notes, or lists of formulas are allowed during the test. You will need a calculator for many of the questions. Some of the problems on the test will ask for exact answers. For such problems, no credit will be given for decimal values produced on a calculator, so be sure you give the required exact answer in these cases. For example, if you are asked for the exact value of $\sec\frac{\pi}{4}$, the correct answer is $\sqrt{2}$, and not $1.414\cdots$. For those problems where a value of a trigonometric function is to be obtained from a calculator, be sure you set the calculator in the correct mode, either degrees or radians. You will have to graph a few functions on the examination. Be sure you label the graphs clearly so that I can tell you know the period, the amplitude, or other important features of the graphed function.

If you haven't done so yet, you should submit the *Request for Examination* form now.

Good Luck on the test!

LESSON 9

INTRODUCTION TO IDENTITIES

READ:

Chapter 4, Sections 1 and 2

INSTRUCTIONAL NOTES:

Algebraic equations come in two flavors. An equation such as $2x+1=7$ is true for some values of x and false for other values of x. In fact, it is true only for $x=3$. On the other hand, the equation $(x+1)^2 = x^2+2x+1$ is true for every value of x. The equation $2x+1=7$ is called a *conditional equation.* Faced with a conditional equation, the problem of interest is to determine the values for which the equation is true. On the other hand, $(x+1)^2 = x^2+2x+1$ is called an *identity.* When presented with a purported identity, the problem of interest is to *prove* the identity is correct. In the case of the identity $(x+1)^2 = x^2+2x+1$, we could show it is correct simply by starting with the left hand side and multiplying it out. The steps would go as follows:

$$\begin{aligned}
(x+1)^2 &= (x+1)(x+1) \\
&= (x+1)x + (x+1)1 \\
&= xx + 1x + x1 + 1\cdot 1 \\
&= x^2 + x + x + 1 \\
&= x^2 + 2x + 1.
\end{aligned}$$

There are a few things that should be noted in this proof. First, we used various rules of algebra that we know, such as the meaning of the exponent 2 in the first line and the distributive property (see page 450 of the text) in the second and third lines. Generally speaking, the steps in such a proof will each be justified based on some algebraic fact that we already know to be correct. A second, more subtle feature of the proof is that we started with one side of the identity and, by making use of valid steps, converted it to the other side. **That is the style that is to be used to verify identities.** There is sometimes a temptation to work with both sides of the identity or to move quantities from one side to the other or to multiply or divide both sides by some quantity or to take the square or square root of both sides. **Resist all such temptations!** Also, do not feel an identity has been verified if the derivation leads to an equation known to be true. That does not mean the initial equation is always true. For example, consider the following *identity* (which isn't really an identity at all, of course): $1 - x = x - 1$. Here is an alleged proof:

$$\begin{aligned} 1 - x &= x - 1 && \\ (1-x)^2 &= (x-1)^2 && \text{squaring both sides} \\ 1 - 2x + x^2 &= x^2 - 2x + 1 && \text{expanding the squares} \\ 0 &= 0 && \text{cancelling.} \end{aligned}$$

Since we have arrived at a true equation, we conclude the original equation is always true. In other words $1 - x = x - 1$ is an identity. **WRONG! of course**.

The lesson to be learned is that when verifying an identity, begin with one side and convert it into the other side. Usually it is a good idea to convert the more complicated looking side into the simpler looking side. There are really only two basic tools available for getting from one step in the derivation to the next. First, legal algebra can be used. For example, if we reach an expression of the form $\frac{(x+2)(x-3)}{5(x-3)}$ we know that we can cancel to produce $\frac{x+2}{5}$. The other legal operation is replacement; that is, we can replace one expression by something known to be equal to it. For example, if we were working with trigonometric functions, and reached an expression containing $\sin^2 x + \cos^2 x$, we can replace that by 1 since we already know that, for any x, $\sin^2 x + \cos^2 x = 1$.

There was a slight lie above. An identity does not have to be true for every value of the variables involved. It needs to be true for every value of the variables *for which both sides of the equation makes sense.* Thus, for example, $\frac{(x+2)(x-3)}{5(x-3)} = \frac{x+2}{5}$ is an identity even though the left side is undefined when $x = 3$, while the right side has value 1 when $x = 3$. As long as the equation is correct when both sides are defined, the equation is called an identity.

We are mostly interested here in identities involving trigonometric function. A short list of basic identities we have already learned is given in the table on pages 218-219 of the text. Be sure you know these so you can use them to verify more complicated identities.

For example, let's verify the identity $\frac{\sin^2 x}{\cos x} + \cos x = \sec x$.

$$\begin{aligned} \frac{\sin^2 x}{\cos x} + \cos x &= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} \\ &= \sec x. \end{aligned}$$

In the first step we used a little algebra:

$$\cos x = \frac{\cos x}{1} = \frac{\cos x}{1} \cdot \frac{\cos x}{\cos x} = \frac{\cos^2 x}{\cos x}.$$

In the second step, we added the two fractions. In the third step we used substitution, applying the known identity $\sin^2 x + \cos^2 x = 1$. Finally, in the last step, we used the identity $\frac{1}{\cos x} = \sec x$.

One use of trigonometric identities is finding the <u>exact</u> evaluation of trigonometric functions. Here is an example of the procedure. Suppose we know that $\tan x = 3$ and that $\sin x < 0$. We can determine the values of the six trigonometric functions of x. First notice that since $\tan x$ is positive and $\sin x$ is negative, x is associated with an angle in the third quadrant. That tells us the sign of the six trigonometric functions of x. To determine the specific values, we'll use some identities. Since $\sec^2 x = 1 + \tan^2 x = 1 + 3^2 = 10$, we see that either $\sec x = \sqrt{10}$ or $\sec x = -\sqrt{10}$. In the third quadrant, the secant is negative, and so we conclude $\sec x = -\sqrt{10}$. Using the identity $\sec x = \frac{1}{\cos x}$, we see $\cos x = -\frac{1}{\sqrt{10}}$. Next, we can determine the value of $\sin x$ by using the identity $\sin^2 x + \cos^2 x = 1$. That tells us that $\sin^2 x + \frac{1}{10} = 1$, so that $\sin^2 x = \frac{9}{10}$. Hence $\sin x = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$ or $\sin x = -\frac{3}{\sqrt{10}}$. Since we were told that $\sin x < 0$, we conclude that $\sin x = -\frac{3}{\sqrt{10}}$. Next, using the identity $\csc x = \frac{1}{\sin x}$, we get $\csc x = -\frac{\sqrt{10}}{3}$. Finally, using $\cot x = \frac{1}{\tan x}$, we see $\cot x = \frac{1}{3}$. Notice that we were able to find all these values without ever actually computing the value of x or drawing triangles!

Suggested Warm-up Assignment:

Section 4.1: 9,25,41.

Section 4.2: 9,25,29,41,47,49,57.

Problems to Submit:

Section 4.1: 10,26,42.

Section 4.2: 11,26,30,42,48,50,58.

LESSON 10

SUM, DOUBLE ANGLE, AND HALF ANGLE IDENTITIES

READ:

Chapter 4, Sections 3 and 4

INSTRUCTIONAL NOTES:

The list of basic identities is extended with the useful

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

which holds for any two numbers x and y. The proof of this identity, just as for the proofs of the other basic identities, is geometric rather than by the methods of the last section where we used the basic identities to verify more complex identities. Carefully study the derivation for the *cosine of a difference* identity presented in the text on pages 235-236.

All the other identities of these two sections are derived from the cosine identity above, so you can see it really is honest to call it a basic identity! The consequences of that one identity are listed in the tables in the text on pages 239, 247, and 249. You will need to follow the steps used to derive each of these identities because you will have to use the same techniques yourself when doing the homework assignment. Don't view the identities as a list of formulas to memorize. Instead, view each one as a new identity to verify. In fact, most people do not keep all these identities in their heads at all times. In addition to the basic identities of the last lesson, the ones from these two sections that would be wise to know cold are:

$$\sin(x \pm y) = \cos x \sin y \pm \cos y \sin x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Note the pattern of signs in these two equations.

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Those are the six new identities that are most often used. If you understand the derivations in the text, you can re-derive the others *on-the-fly* when you need them.

Using these new identities, it is possible to compute the exact values of many numbers besides the usual 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. For example, here is one way to find the exact value of $\cos\frac{\pi}{12}$ using the basic identity of this lesson. We begin by noting that $\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$. So

$$\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\
&= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\
&= \frac{1+\sqrt{3}}{2\sqrt{2}}.
\end{aligned}$$

For another example, let's use a half angle formula from section 4.4 to find the exact value of $\sin\frac{\pi}{8}$.

$$\begin{aligned}
\sin\frac{\pi}{8} &= \sin\frac{1}{2}\cdot\frac{\pi}{4}\\
&= \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}\\
&= \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}\\
&= \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}\\
&= \sqrt{\frac{2-\sqrt{2}}{4}}\\
&= \frac{\sqrt{2-\sqrt{2}}}{2}.
\end{aligned}$$

Note that the positive square root was used in the first step since $\frac{\pi}{8}$ is associated with a first quadrant angle, and so the sine of $\frac{\pi}{8}$ will be positive.

As another example of the use of these identities, let's determine the exact value of $\sin(x+y)$ if we are told that x is a first quadrant value with $\cos x = \frac{1}{3}$, and y is a second quadrant value with $\cos y = -\frac{1}{2}$. Using the identity for the sine of the sum of two values, we get

$$\sin(x+y) = \sin x \cos y + \cos x \sin y = (\sin x)\left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right)(\sin y).$$

We now need to determine the values of $\sin x$ and $\sin y$. Using the identity $\sin^2 t + \cos^2 t = 1$, which is true for all values t, and keeping in mind that the sine is positive for both first and second quadrant values, we see

$$\sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\sin y = \sqrt{1-\left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Substituting these values for $\sin x$ and $\sin y$ in the earlier equation, we get

$$\begin{aligned}\sin(x+y) &= (\sin x)\left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right)(\sin y)\\ &= \left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{6}\\ &= \frac{\sqrt{3}-2\sqrt{2}}{6}.\end{aligned}$$

Concerning the double angle identities in the table on page 245, don't make the error of incorrectly *factoring out* the coefficient of x. For example, don't try manipulations like $\sin 2x = 2\sin x$ or $\cos 3x = 3\cos x$. Such manipulations are **WRONG!** As an example to show the first one is not an identity, consider the number $x = \frac{\pi}{4}$. For this value of x, we see $\sin 2x = \sin\frac{\pi}{2} = 1$ while $2\sin x = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}$, and so the two sides of the so-called equation are certainly not equal. In general, there is no easy way to deal with coefficients in front of the variable inside a trigonometric function. The double angle formulas show how to convert $\sin 2x$ and $\cos 2x$ into simpler quantities, and with a bit of work, we can handle other integers in place of the 2. For example, here is a way to deal with $\sin 3x$. The idea is to use the identity for the sine of a sum, noting that $3x = 2x + x$.

$$\begin{aligned}\sin 3x = \sin(2x + x) &= \cos 2x \sin x + \sin 2x \cos x \\ &= \left(\cos^2 x - \sin^2 x\right) \sin x + (2 \sin x \cos x) \cos x \\ &= \left((1 - \sin^2 x) - \sin^2 x\right) \sin x + 2 \sin x \cos^2 x \\ &= \left(1 - 2\sin^2 x\right) \sin x + 2 \sin x \left(1 - \sin^2 x\right) \\ &= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x \\ &= 3 \sin x - 4 \sin^3 x.\end{aligned}$$

That establishes the identity $\sin 3x = 3 \sin x - 4 \sin^3 x$.

Suggested Warm-up Assignment:

Section 4.3: 23,31,39,43,51.

Section 4.4: 13,21,25,35,61.

Problems to Submit:

Section 4.3: 24,32,40,44,52.

Section 4.4: 14,22,26,36,62.

LESSON 11

INVERSE TRIGONOMETRIC FUNCTIONS

READ:

Chapter 5, Section 1

INSTRUCTIONAL NOTES:

The question, *What values of* y *have a cosine equal to* $\frac{1}{2}$*?*, has infinitely many answers. $y = \frac{\pi}{3}$ is one answer, but since the cosine function is 2π-periodic, $y = \frac{\pi}{3} + 2k\pi$ will also be an answer, where k is any integer. In other words, if $\cos y = \frac{1}{2}$, then y can be any of the values

$$\ldots, \quad \frac{\pi}{3} - 3 \cdot 2\pi, \quad \frac{\pi}{3} - 2 \cdot 2\pi, \quad \frac{\pi}{3} - 2\pi, \quad \frac{\pi}{3}, \quad \frac{\pi}{3} + 2\pi, \quad \frac{\pi}{3} + 2 \cdot 2\pi, \quad \frac{\pi}{3} + 3 \cdot 2\pi, \quad \cdots$$

Or more neatly,

$$\ldots, \quad -\frac{17\pi}{3}, \quad -\frac{11\pi}{3}, \quad -\frac{5\pi}{3}, \quad \frac{\pi}{3}, \quad , \frac{7\pi}{3}, \quad \frac{13\pi}{3}, \quad \ldots$$

There are even more possible values of y since $\cos \frac{5\pi}{3} = \frac{1}{2}$, and so $y = \frac{5\pi}{3} + 2k\pi$, k any integer, will also be answers to the question. If the two sets of solutions are combined, we can see that all possible answers to the question will be given by $y = \frac{(2k+1)\pi}{3}$, where k is any integer. Be sure you write that list of y's out to see that that formula produces all the y's given above. The phrase *where k is any integer* is often abbreviated by the symbols $k \in \mathbf{Z}$. Note: $\mathbf{Z}$ is the symbol for the set of all integers, and $k \in \mathbf{Z}$ is read as k is in $\mathbf{Z}$. So we would say that the solutions to $\cos y = \frac{1}{2}$ are $y = \frac{(2k+1)\pi}{3}, \quad k \in \mathbf{Z}$.

Of all these values of y, one is singled out as the *principal solution.* In general, for any number x between -1 and 1, the principal solution to the question of which y's have $\cos y = x$, is the y that occurs between 0 and π. If you look at the graph of the cosine function, page 278 of the text, you will see there will always be *exactly one* y between 0 and π that will have a cosine equal to such a given value of x. The principal solution y is denoted by the symbol $\cos^{-1} x$, so we would write $y = \cos^{-1} x$. In the example above,

$\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. It would be wrong to write $\cos^{-1}\frac{1}{2} = \frac{5\pi}{3}$, even though $\cos\frac{5\pi}{3} = \frac{1}{2}$, because $\frac{5\pi}{3}$ in not in the interval allowed for the principal solution. **Warning:** The -1 attached to the cosine in that symbol has no connection what-so-ever with reciprocals. It is simply an unfortunate historical accident that the -1 is used to denote both reciprocals of numbers and principal values described above. Some books, to avoid this confusion, use $\arccos\frac{1}{2}$ in place of $\cos^{-1}\frac{1}{2}$. Note the following two equations. Be sure you understand why they are both correct.

$$\cos^{-1}\frac{1}{2} = \frac{\pi}{3} \qquad \left(\cos\frac{1}{2}\right)^{-1} = \frac{1}{\cos\frac{1}{2}} = \sec\frac{1}{2} = 1.139\cdots$$

In general, if $x > 1$, then $\cos^{-1} x$ is undefined since there are no values of y for which $\cos y$ will be bigger than 1. Likewise, $\cos^{-1} x$ is undefined if $x < -1$. Note that if $-1 \le x \le 1$, then $\cos(\cos^{-1} x) = x$, since $\cos^{-1} x$ is a value whose cosine is equal to x, and so its cosine will equal x! Asking for the value of $\cos(\cos^{-1} x)$ is sort of like the classic question, *Who's buried in Grant's tomb?* The question itself tells you the answer. Note also that $\cos^{-1}(\cos x) = x$ is true, provided $0 \le x \le \pi$. For example, $\cos^{-1}(\cos\frac{\pi}{3}) = cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. But don't make the error of writing $\cos^{-1}(\cos x) = x$ for other values of x, since that would be wrong. For example, $\cos^{-1}(\cos 2\pi) \neq 2\pi$. In fact, $\cos^{-1}(\cos 2\pi) = cos^{-1} 1 = 0$. Remember: $\cos^{-1}(\cos x)$ equals that number <u>between 0 and π</u> with the same cosine as x.

The entire story above can be repeated for the sine and tangent functions. For the sine, the principal solution to $\sin y = x$ is denoted by $y = \sin^{-1} x$, and the y is required to be in the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. If you look at the graph of the sine function, you will see that for each x between -1 and 1, that interval produces exactly one y with $\sin y = x$.

For the tangent, the principal solution to $\tan y = x$ is denoted by $y = \tan^{-1} x$, and y is required to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$y = \cos^{-1} x$ is called the *inverse cosine function*, and the other two just described are the inverse sine and inverse tangent functions respectively. The inverse secant, inverse cosecant, and inverse cotangent functions can be defined in the same sort of way, but these three occur rarely in real life, so we won't pursue them. If you ever need to know about them, the facts are presented in section 5.2.

Your calculator automatically produces the correct (in other words, principal solution) for each of the inverse trigonometric functions, so you don't have to think about it. However, when exact values are needed, it will be necessary to think carefully about the interval in which the principal solution of each inverse trigonometric function must lie.

Suggested Warm-up Assignment:

Section 5.1: 5,9,19,23,25,35,41,61,63.

Problems to Submit:

Section 5.1: 6,10,20,24,26,36,42,62,70.

LESSON 12

TOUGHER TRIGONOMETRIC EQUATIONS

READ:

Chapter 5, Section 3

INSTRUCTIONAL NOTES:

Now that the process for finding principal values is mastered, the problems in section 5.3 return to the question posed at the start of this lesson: Finding all the values for which a trigonometric equation is correct. In some of the examples and in some of the exercises at the end of section 5.3, you are asked to find all solutions in a certain interval to a trigonometric equation. I would suggest that instead, you always find all solutions to the equation, and only then look at the interval where the solutions are to appear. It is an easy matter to pick the ones that are wanted from the list of all solutions.

Correct solutions to trigonometric equations can often be expressed in several different ways. For example, if we want all solutions to $\cos y = \frac{1}{2}$, then as we saw above, the solutions are given by

$$y = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad y = \frac{5\pi}{3} + 2k\pi, \qquad k \in \mathbf{Z}.$$

With a little bit of thought, you can see that the solution could just as well be written as

$$y = \frac{(2k+1)\pi}{3}, \qquad k \in \mathbf{Z}.$$

So, just because your answer does not look like the one in the text, do not conclude that you are wrong (or the answer in the book is wrong!). Instead, see if the two solutions are really the same despite the apparent differences.

$2\sin^2 x + \sin x - 1 = 0$ is an example of a trigonometric equation. To solve such an equation means to determine all the values of x for which the equation is correct. While there is no sure fire step by step procedure that will produce the solutions to a trigonometric equation, there is a list of suggestions on page 297 of the text that can help get you started. There is no doubt that solving trigonometric equations can be very tricky. The problems in this lesson are probably the most difficult in the course. All the main techniques are well demonstrated in the textbook examples, so there isn't much more I can add. Study the section carefully before attempting the homework problems.

Warning: These problems tend to take quite a bit of time. Don't rush through them. Savor each one! Be sure to check your answers in the original equation when you finish each problem, both to make sure no major error has been made and to weed out extraneous solutions. Since solving a trigonometric equation often involves changing the form of the equation using identities or algebraic manipulations (such as squaring both sides of an equation), it is always possible that some manipulations will introduce extraneous solutions. You need to always check for that possibility.

The assignment below should be done without a calculator. Give exact answers. As in the last lesson, I suggest you give all solutions to each problem, even if only solutions in a specified interval are requested.

Suggested Warm-up Assignment:

Section 5.3: 1,5,9,15,17,21,23,29.

Problems to Submit:

Section 5.3: 2,6,10,16,18,22,24,40.

LESSON 13

LAW OF SINES AND LAW OF COSINES

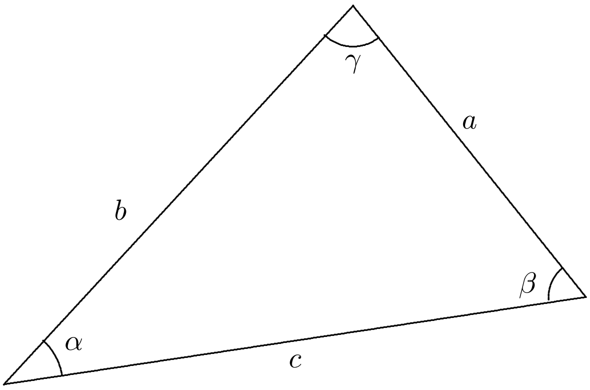
READ:

Chapter 6, Sections 1 and 2

INSTRUCTIONAL NOTES:

Appropriately for trigonometry, with this final lesson we come full circle, returning to the problem of solving triangles with which we began the course. Back then we practiced solving right triangles. Now we will learn to solve arbitrary triangles. Traditionally degree measure is used when solving triangles, so it is going to be necessary to make sure the calculator (and the mind!) is set in degree mode for these two sections.

There are two basic tools for solving general triangles. Using the standard notation of the diagram below,



the two laws are

(1.) The Law of Sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.

(2.) The Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

Notice that if $\alpha = 90°$, then the Law of Cosines says $a^2 = b^2 + c^2 - 2bc\cos 90° = b^2 + c^2 - 2bc \cdot 0 = b^2 + c^2$. In other words, in this case, $a^2 = b^2 + c^2$, which is the Pythagorean relation for a right triangle. Thus the Pythagorean Theorem is a special case of the Law of Cosines. Be sure to study the derivations of these two laws on pages 331 and 343 of the text.

In solving a triangle, we will be given three of the six quantities a, b, c, α, β, and γ, and our job will be to determine the remaining three. At least one of the three given items must be a side since if we are told just the three angles the lengths of the sides cannot be determined.

The Law of Cosines is best suited to solving triangles

A. for which we know two sides and the angle included between them (Example 1, page 345).

B. for which we know all three sides (Example 2, page 347).

The Law of Sines is the tool of choice for solving triangles

C. for which we know one side and two (and hence all three) angles (Examples 1 and 2, pages 333 and 334).

D. for which we know two sides and the angle opposite one of the two sides.

The last case (D.) is the so-called *side-side-angle* or SSA case, and it is a little different from the first three cases listed above. For problems of type A and C, there will always be exactly one triangle that will have the three known measurements. For problems of type B, normally there will be exactly one solution as well; but it can happen there are no solutions to a type B problem. For example, clearly there is no triangle with side lengths 1, 1, and 20. Try to draw such a picture and you will see why. In general, given three proposed side lengths for a triangle, any two of the lengths must add up to more than the third length if there is to be a solution. So it is pretty clear when a type B problem will have a solution and when it won't. But for problems of type D, the story is more involved. For example, if we are given measurements $a = 1$, $b = 20$, $\alpha = 30°$, it's pretty clear that <u>no</u> triangle is actually possible since $a = 1$ is way too short to reach side c no matter what angle we put in for γ. On the other hand, if the length of a is just right (pun intended!), so that it gives the perpendicular distance from the top vertex to side c, then there will be exactly one (right) triangle which will have the known measurements. If the given side a is just a little longer yet, then there will be two triangles, and if a is still longer, there will only be one triangle with the known measurements. In general, if the known angle α is an acute angle, there will be either 0, 1 or 2 triangles with the given measurements. If the given angle α is obtuse (exceeds 90°) then there will be either 0 or 1 triangles with the given measurements. All these possibilities are shown in the table on page 335 of the text.

Generally speaking, which of the five possibilities actually occurs in a given SSA problem becomes clear during the solution process. For example, if there are no solutions to a certain problem, then at some stage of the solution an angle might appear with a sine bigger than 1. Since that is impossible, we conclude the problem has no solutions.

The two cases that are hard to distinguish are the one labeled (c) and (d) in the table. Here is how to always get the correct solutions (case c) or solution (case d). Let's suppose a, b, and α are the given measurements. In the course of solving the problem the quantity $\sin\beta$ will be computed, and in cases c and d it turns out that $\sin\beta < 1$. Using the $\sin^{-1}$ button on a calculator, the angle β can now be found. This β always provides one solution triangle. Next we check to see if there is a second possible triangle. Once β has been computed always calculate $180° - \beta$ which we will call β_1. Note that β_1 will give the angle opposite side b in the second possible solution. Now it is easy to check to see which of the cases, c or d, occurs as follows. Add α and β_1. If this sum is 180° or more, then clearly, no second triangle is possible, and case d has occurred. But if the sum is less than 180°, then case c has occurred and β_1 will be the angle opposite side b in the second solution.

Remember: SSA = Sometimes Solution Ambiguous. Whenever the unknown angle found opposite the known side is less than 90°, it is necessary to check if there is a second solution with that angle replaced by its supplement.

Examples 3, 4, and 5, pages 336-337 of the text show the method of solution for the various possibilities.

Suggested Warm-up Assignment:

Section 6.1: 3,11,13,19,21.

Section 6.2: 3,9,15,25,29.

Problems to Submit:

Section 6.1: 4,12,14,20,22.

Section 6.2: 4,10,16,26,30.

The next lesson is the final examination. You may submit the *Request for Examination* form with this assignment if you wish. See the next lesson for some information about the test.

LESSON 14

FINAL EXAMINATION

ABOUT THE TEST:

The examination will deal directly with only the material covered since the midterm examination: Chapter 4, sections 1 trough 4, Chapter 5, sections 1, 3, and sections 1 and 2 of Chapter 6.

There will be 14 questions on the test. You will have $2\frac{1}{2}$ hours for the examination, but it probably won't take you that long if you are well prepared. The main topics covered are solving trigonometric equations, verifying identities, and solving triangles. Some problem will ask you to use the half-angle and double angle identities to compute values of trigonometric functions. The problems will all be very much like the homework problems you have been submitting. There will be no surprises!

To prepare for the examination, you should go over the homework that has been graded and returned to you. Pay particular attention to corrections that appear on your homework. If you want more practice, each chapter in the text is followed by a section of *Review Problems* that might be worth looking over. The Review Section problems have complete solutions written up in the Student Solutions Manual.

No books, notes, or lists of formulas are allowed during the test. You will need a calculator for a few of the questions. Most of the problems on the test will ask for exact answers. For such problems, no credit will be given for decimal values produced on a calculator, so be sure you give the required exact answer in these cases. For example, if you are asked for the exact value of $\sec\frac{\pi}{4}$, the correct answer is $\sqrt{2}$, and not $1.414\cdots$. You will have to solve four trigonometric equations on the test. Be sure you give all the solutions to such equations.

If you haven't done so yet, you should submit the *Request for Examination* form now.

Good Luck on the final!