UNIVERSITY OF NORTH DAKOTA
DEPARTMENT OF CORRESPONDENCE STUDY
DIVISION OF CONTINUING EDUCATION
Grand Forks, North Dakota

MATH 102
INTERMEDIATE ALGEBRA

Lessons: 19
Credit Hours: Three (3)
S/U grading only

TEXTBOOK: Aufmann, Barker, Lockwood
Algebra: Introductory and Intermediate.
Houghton Mifflin Publishing Company.
4^{th} Edition. 2007. ISBN: 0-618-60953-9

OPTIONAL: *Student Solution Manual.*

Course Author: Jerry Metzger, Professor of Mathematics 4/2006

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INTRODUCTION

Suitable background for this course is a year of high school algebra. It is assumed you have successfully completed such a course. Some chapters in the text book for this course are covered in the Elementary Algebra Correspondence course (Math 100). In particular, you should be at least a little familiar with the topics covered in Chapters 1, 2, 6, and 7 of the text. Of those chapters, Chapter 7 on factoring is likely to be the most troublesome. Factoring will be reviewed briefly at the start of lesson 6, just before it is needed, but you might find it necessary to do more reviewing of that material on your own.

There are two main goals of intermediate algebra. The first is to review and extend the manipulative skills you learned in elementary algebra, and the second is to provide the tools you will need to understand finite math, trigonometry, or college algebra.

Your job for each lesson will be to read over the assigned sections in the text and in these notes. Be sure you can follow all the details of each example presented. As you read along, have a pencil and a piece of paper handy so you can fill in any missing steps in the examples. Don't be in a hurry. Each of the lessons covers about the amount of material that would be normal for one week in a regular class (in other words, about $2\frac{1}{2}$ hours of class time). So expect to spend a reasonable amount of time reading over the material for each lesson.

There is a student solution manual that has all the odd numbered problems in the text worked out in some detail. You might want to get a copy of that if you find the number of examples in the text is a little too meager for your needs. Also, there are a number of inexpensive self-help books available at most bookstores that are good sources of more examples. Particularly good is the *Schaum's Outline* series. Each book costs about \$16. *Elementary Algebra* has ISBN $0-07-052262-6$, and *Intermediate Algebra* has ISBN $0-07-060839-3$.

For each lesson, you should do pretty much all the odd numbered problems. These have answers in the back of the book, so you can check your work. Once you have solved enough of the odd numbered problems so that you feel confident about the material, it is time to tackle **Problems to Submit**. Expect to spend an hour or so on these problems for each lesson. The **Problems to Submit** will consist of a small number of the even numbered problems that will allow me to check your progress. Write the solutions up in a neat and logical fashion. An important part of algebra is learning how to write in a mathematically mature way, so take your time and think about what you are writing. You must show all the work leading to the answer so that the steps can be easily followed. A correct answer without correct reasoning will earn very little credit. Be particularly careful working on the Problems to Submit since you won't have answers in the back of the book to guide you.

If you have access to a computer and the Internet, you might want to consider submitting your assignments directly through Blackboard as attachments. Details for generating the .pdf files and attaching your homework problems are on found under the Syllabus link in the Blackboard site for the course. Assignments submitted via the Internet will normally be graded and returned within two to three days. Alternatively, you may mail paper versions of the written assignments to the Department of Correspondence & Online Study. They will be graded and returned to you. A turn-around time of 15 days or so seems to be typical for assignments sent via the post office. All grades will be posted in Blackboard once assignments are graded.

There will be 16 written assignments. Each assignment will be graded on a scale from 0 to 100, and an overall average of the 16 assignments will be computed. That average will count as $\frac{1}{4}$ of your final grade.

If you have any questions about the reading, the examples, the homework problems, or the instructional notes provided here, you can send your questions to me at jerry.metzger@und.edu
I will normally answer e-mail within two days.

GRADING SYSTEM

There will be a total of 400 points distributed as follows:

Lesson 7: Examination I - Lessons 1-6:	100 points
Lesson 13: Examination II - Lessons 8-12:	100 points
Lesson 19: Examination III - Lessons 14-18:	100 points
Average of 16 written assignments:	100 points

This course is pass-fail grading (S/U) only. Your grade will be based strictly on the total number of points you earn.

Grades will be determined as follows:

270 - 400 total points - Final grade of S

0 - 269 total points - Final grade U

Examinations will consist of problems very much like the problems assigned for homework. Use of notes and books is not permitted during examinations. Also, on the examinations, calculators will not be needed and will not be allowed. It's probably a good idea to avoid using calculators for homework as well.

LESSON 1

The Coordinate Plane

READ:

Sections 4.1.A,B,C

Read the Introduction to these notes.

INSTRUCTIONAL NOTES:

You have almost certainly seen graphs of equations in the x,y-plane (or Cartesian plane) before, but you may have forgotten some of the terminology associated with the plane, so review those words at the beginning of section 4.1A. For example, you probably recall that given an ordered pair $P = (2, 7)$, to **plot** the point P means to put a dot on the Cartesian plane at the spot with coordinates 2 and 7. The first coordinate of the point P is 2. It is called the x-coordinate. The second coordinate of P is 7. It is called the y-coordinate.

An **equation** is a mathematical sentence that asserts the equality of two expressions. For example, $3x + 1 = 16$ is an equation involving the unknown x. A **solution** to an equation is a value of the variable for which the equation becomes a true statement. So in the example above, $x = 5$ is a solution to the equation since it is true that $3 \cdot 5 + 1 = 16$. On the other hand, $x = 2$ is not a solution since $3 \cdot 2 + 1 = 16$ is false.

In section 4.1B, we are going to be interested in equations involving two variables, x and y. Consider an equation such as $y = 3x + 1$. Note that if $x = 3$ and $y = 10$, the equation is true. We will say the $(3, 10)$ **satisfies** that equation, or that $(3, 10)$ is a **solution** of the equation. On the other hand, $(2, 5)$ does not satisfy the equation, since $5 = 3 \cdot 2 + 1$ is false. There are lots of ordered pairs that satisfy the equation. Here are some more: $(5, 16)$, $(\frac{1}{3}, 2)$, $(0, 1)$, and $(-2, -5)$. In fact, there are infinitely many ordered pairs that satisfy that equation.

It is often handy to have a picture of the solutions of an equations. Such a picture can be produced by plotting the pairs that satisfy the equation in the x, y-plane. This picture is called the **graph** of the equation.

Problems to Submit

4.1.A: 6, 10

4.1.B: 18, 22, 26, 28, 32, 36, 38

4.1.C: 40

LESSON 2

Lines in the Plane

READ:

Sections 4.2.A, and 4.3.A,B,C

INSTRUCTIONAL NOTES:

A **relation** between two sets is any rule which says that certain members of the first set are associated with certain members of the second set. An example will help clarify that definition.

Here is a non-mathematical example: Let A be the words in English, and let B be the alphabet of 26 letters. We will associate a word with a letter if that letter occurs in the word. That means we will associate **cat** with **c** and also with **a** and also with **t**, but we will not associate **cat** with **m**. To indicate **cat** and **c** are associated, we will write them as an ordered pair: $(\mathbf{cat}, \mathbf{c})$, and we will say that the pair is in the relation we are considering. Likewise $(\mathbf{cat}, \mathbf{a})$ and $(\mathbf{cat}, \mathbf{t})$ are in the relation, but $(\mathbf{cat}, \mathbf{m})$ in not. Notice that letter **c** also appears in lots of other pairs in the relation: $(\mathbf{cook}, \mathbf{c})$, $(\mathbf{ace}, \mathbf{c})$, $(\mathbf{basic}, \mathbf{c})$, and so on.

Here is a more mathematical example. We can form the relation of all pairs (x, y) such that $x^2 + y^2 = 25$. Some pairs in this relation are $(0, 5)$, and $(0, -5)$, and $(3, 4)$, and $(-3, 4)$, and $(3, -4)$, and so on. However, the pair $(2, 6)$ is not in this relation since $2^2 + 6^2 = 40$ and not 25.

For a mathematical relation, such as the one in the previous paragraph, we can draw a graph. We put a dot in the Cartesian plane for each pair in the relation. If you actually do that for the last example, the graph will be a circle of radius 5 with its center at $(0, 0)$.

Here's one last mathematical example. We will say (x, y) is a pair in our new relation provided $y = 2x + 1$. Some pairs in this relation are $(1, 3)$, and $(-3, -5)$, and $(\frac{1}{2}, 2)$. You might recognize the graph of this relation as a straight line. Straight lines will be our main interest for a while.

There is an important difference between the first two examples above, and the third example. In the first two examples of relations, it was possible for a first coordinate of a pair to be associated with more than one second coordinate. For example, $(\mathbf{cat}, \mathbf{c})$ and $(\mathbf{cat}, \mathbf{a})$ are both in the first relation. Also, $(3, 4)$ and $(3, -4)$ are both in the second. But in the third relation, each possible first coordinate is associated with only one second coordinate. This difference is important enough to deserve a name. If, in a relation, each possible first coordinate is associated with exactly one second coordinate, then the relation is called a **function**.

The **domain** of a relation consists of all the different first coordinates of pairs in the relation, while the **range** comprises all the second coordinates. In the *words* example above, the domain consists of all English words, and the range consists of the 26 letters of the alphabet. In the third example (the straight line), both the domain and the range consist of all real numbers.

There is some special notation introduced when dealing with functions. First of all, functions are usually given a name, and the name is often a single letter such as f or g or h or F. In the straight line example above, we might call that function L. Names are often selected to help us remember something about the function. In this case, L reminds us of the word *line*. In other mathematics courses, function names that are several letters long are sometimes selected. For example, in trigonometry, there is a function called **tan** which is short for the word *tangent*.

There is a special use for the name of a function. Suppose there is a function f, and that x is a member of the domain of f. Then the symbol $f(x)$ stands for the second coordinate of the ordered pair in f whose first coordinate is x. Back to the straight line example above, $y = 2x + 1$, which we decided to call L. In that example $L(4) = 9$, since the pair $(4, 9)$ is in the relation L. Sometimes the function is written as $L(x) = 2x + 1$. The right side really amounts to a rule to tell you how to find the second coordinate that goes with a first coordinate of x in the function L. So $L(6) = 2 \cdot 6 + 1 = 13$, and $L(-2) = 2 \cdot (-2) + 1 = -3$.

In that same function L, what would be the second coordinate of the pair with first coordinate a (where a is some number)? The answer is $L(a) = 2a + 1$. And what is the second coordinate that goes with $a + b$? Answer: $L(a + b) = 2(a + b) + 1 = 2a + 2b + 1$.

There are certain functions that come up often enough to deserve their own names. In 4.3A and B, we look at functions of the form $f(x) = mx + b$ (where m and b are particular numbers). These are called **linear functions**. The graph of a linear function is a straight line.

Notice that the linear equation $y = 3x + 1$ could be rewritten as $-3x + y = 1$. Any equation that looks like $Ax + By = C$, where A, B, and C are some constants (A, B, not both equal to 0), will have a straight line graph, and every straight line is the graph of such an equation. So, for example, $4x + 3y = 11$ will have a graph that is a straight line, as will $15x - 19y = 88$, and so on.

The linear equation $y = 3$ is satisfied by $(2, 3)$, $(-1.7, 3)$, $(0, 3)$, and in fact by every ordered pair with second coordinate 3. If you plot those points, you get a horizontal line through 3 on the y-axis. Likewise, any equation of the form $y = a$ will have a horizontal line for its graph. Equations of the form $x = a$ have vertical line graphs.

Problems to Submit

4.2.A: 4, 16, 20, 38, 62

4.3.A: 6, 8

4.3.B: 12, 16

4.3.C: 20, 22

LESSON 3

Slopes of Lines and Equations of Lines

READ:

Sections 4.3.D, 4.4.A,B, and 4.5.A,B

INSTRUCTIONAL NOTES:

Application problems, also called word problems or story problems, are an important part of mathematics. In fact, it is probably fair to say that solving application problems is the main reason most people are interested in mathematics. After all, it is pretty rare out in the real world to bump into a problem like *Simplify* $3x^2y^3 + 7x^2y^3$. Instead, you might want to determine the height of a mountain say by measuring its shadow, setting up an equation about the height, and applying mathematical methods to solve the equation. This section of the text provides practice with this sort of applied problem. The ability to solve applied problems is very valuable.

Applied problems tend to be quite a bit more difficult than the problems in the section on solving equations. In fact, such problems are *one step* harder in the sense that it is first necessary to figure out the equation to work with, and then, using the ideas of the last lesson, to actually graph the equation.

A few hints about solving word problems might be helpful in this lesson, and, even more, in future lesson:

1) If possible, draw and label a diagram. If the problem concerns a rectangle, draw a rectangle and label the items in the rectangle that are known and the ones that are asked for. If the problem concerns a car on a trip from city A to city B make a dot for A and one for B, and a line joining them to represent the road. Label the diagram with information from the problem, such as direction of the car or mileage between the cities. Not all problems can be associated with a diagram, but do not skip this step if a diagram might be at all helpful.

2) Introduce a letter to stand for the unknown quantity to be found and describe the quantity in words. So if the problem asks for the speed of the car, a good start might be to write *Let* $s =$ *the speed of the car*. If the problem is to find the width of a rectangle, perhaps a good place to begin

(after drawing a diagram of course) is to write *Let* $w =$ *the width of the rectangle.*

A good way to make an easy problem difficult is to skip these two suggestions!

One advantage of knowing that the graph of an equation is a straight line is that it is only necessary to plot two points on the line, since two points determine a line. Suppose we wanted to plot the graph of the equation $2x + 3y = 12$. If we set $x = 0$ we note that $y = 4$, so $(0, 4)$ is one point on the graph. If we plug in $x = 3$, we find $y = 2$. So $(3, 2)$ is a second point on the line. That's all we need to plot the line: join the two points $(0, 4)$ and $(3, 2)$ by a straight line and extend it in both directions, and there is the graph of $2x + 3y = 12$.

Straight line graphs have various tilts. There is a number that can be computed for a line which tells us how much the line tilts. The number is called the **slope** of the line. To find the slope of a line, first find two points on the line. For example, the line $2x + 3y = 12$ graphed earlier passes through the points $(0, 4)$ and $(3, 2)$. The slope is defined to be the difference of the y-coordinates divided by the difference of the x-coordinates. The letter m is traditionally used to denote slopes. So this line has slope

$$m = \frac{4 - 2}{0 - 3} = \frac{2}{-3} = -\frac{2}{3}$$

A line with slope 0 is horizontal. A line with a positive slope is heading uphill as you move to the right, while a line with a negative slope is heading downhill. The more positive the slope is, the steeper the line heads uphill. A line of slope 10 is much steeper than a line of slope 2. Likewise, the more negative the slope is the steeper the line heads downhill.

So far, given the equation of a line the goal was to draw the graph of the line, or to find the slope of the line, or to determine some points on the line. Now let's reverse that question. Some information about a line will be given, and the goal will be to determine an equation for the line.

The text describes several different methods for determining a line's equation, but when all the shouting is over, there is really only one way to find an equation of a line, so it is not really necessary to remember all those different forms. The one and only way to find an equation of a line is to somehow be told the slope of the line and a point the line passes through. If we know the slope, m, and a point on the line, say $P = (a, b)$, Then an equation for the line is

$$\frac{y - b}{x - a} = m$$

To see why that is correct, suppose the $Q = (x, y)$ is any point on the line. Since $P = (a, b)$ and $Q = (x, y)$ are two points on the line, the slope can be calculated, as before, as $\frac{y-b}{x-a}$. But we are told the slope is m. These two values of the slope must be equal. And the result is the promised equation: $\frac{y-b}{x-a} = m$. Normally, you would multiply both sides by $x - a$ and simplify to make the equation look nicer, but the hard work has been done.

For example, suppose we want an equation for a line through the point $(2, -5)$ with slope 3. The pattern given above provides the equation

$$\begin{aligned} \frac{y-(-5)}{x-2} &= 3 \\ y + 5 &= 3(x-2) \\ y &= 3x - 11 \end{aligned}$$

DO NOT FORGET: To find an equation for a line, figure out a point $P = (a, b)$ on the line, and figure out the slope m of the line, and write down $\frac{y-b}{x-a} = m$. Then simplify.

Whenever you are asked for an equation for a line, somewhere lurking in the information about the line must be these two items: a point on the line and the slope of the line. Dig those two items out, and you are done. Those two items can be disguised in many different ways, so it may take some thought to determine them, but they are always there somewhere.

Problems to Submit

4.3.D: 30, 32

4.4.A: 8, 18, 24

4.4.B: 34, 40

4.5.A: 6, 28

4.5.B: 46, 52

LESSON 4

Parallel and Perpendicular Lines

Systems of Equations

READ:

Sections 4.5.C, 4.6.A, 4.7.A, and 5.1.A,B

INSTRUCTIONAL NOTES:

There is an important connection between two geometric relations of two lines and the slopes of the lines.

1) The slopes of lines that are parallel are equal.

2) The slopes of perpendicular lines are negative reciprocals of each other.

Here is the meaning of rule 2): If two lines are perpendicular, and one line has slope 3, the other has slope $-\frac{1}{3}$. If one has slope $-\frac{5}{11}$, then the other has slope $\frac{11}{5}$, and so on.

For example, suppose we want an equation for a line through the point $(3, 7)$ which is perpendicular to the line with equation $4x-3y = 11$. Well, there is a point on the line we are looking for $P = (3, 7)$, but the slope isn't given directly. However, we can figure it out. The line $4x - 3y = 11$ has slope $\frac{4}{3}$, and since the line we want is perpendicular to that line, our line must have slope $-\frac{3}{4}$. Now we can fill in the pattern for an equation for the line:

$$\frac{y-7}{x-3} = -\frac{3}{4}$$

$$4(y-7) = -3(x-3)$$

$$4y - 28 = -3x + 9$$

$$3x + 4y = 37$$

A linear inequality is a linear equation in which the = sign has been replaced by $<$ (or $>$, or $\leq$, or $\geq$). So, for example, $x + 2y < 8$ is a linear inequality. The graph of a linear equation is a straight line. The graph of a linear inequality will be all the points in the plane on one side or the other of the graph of the corresponding linear equation. As a particular example, let's solve the inequality $x + y < 2$. Begin by graphing the corresponding linear equation $x + y = 2$. This will be a line of slope -1 with x and y intercepts both equal to 2. (Draw a sketch!) Now the solutions to the linear inequality $x + y < 2$ will be all points on one side or the other of that line. To find out which side, select a test point on one side of the line, plug the values into the inequality, and see if the result is true. If it is, your test point is on the correct side and that side is the solution to the inequality. If the result of plugging in is a false statement, then the selected point was from the wrong side of the line and the other side is the solution. In the example above, suppose we select the test point $(5, 0)$. Plugging those values into the inequality gives $5 + 0 < 2$ which is false. So the correct solution to $x + y < 2$ is all the points on the other (left) side of the line.

In the example of the last paragraph, the line $x + y = 2$ was not part of the solution since for points on the line, $x + y$ is exactly 2, and not less than 2. To indicate that the line is not included in the solution of the inequality, it is drawn in as a broken line. On the other hand, the solution to $x + y \leq 2$, would be the solution we found to $x + y < 2$ together with the line $x + y = 2$. To indicate that the line is included in the solution to $x + y \leq 2$, it is drawn in as a solid line.

Consider the pair of linear equations $\begin{cases} 2x + 3y = 11 \\ 9x - 2y = 3 \end{cases}$. If $x = 1$ and $y = 3$ are plugged into those two equations, then both equations are satisfied simultaneously. In other words, $(1, 3)$ is a common solution to that *system of two equations.* Geometrically, the meaning is this: if both equations are graphed, the picture will be two straight lines, and those lines meet at the point $(1, 3)$. We will call $(1, 3)$ a solution to the system. More generally, the solution to a system of equations is any point that is part of the graph of every equation in the system. In other words, the solutions to a system are the points where all the graphs meet each other.

The system above is called a system of two linear equations in two unknowns. If you think about the various possibilities for the graphs of straight lines in the Cartesian plane, you will see that there are only three events that might occur:

1) The lines are distinct and parallel. In this case, there are no points on both lines, and the system has no solutions. The system is called *inconsistent.*

2) The lines are not parallel. In this case, there will be exactly one point where the lines cross and so the system will have exactly one solution. The system is called *independent.*

3) The two lines have exactly the same graph (for example the system might be made up of $x + y = 1$ and $2x + 2y = 2$, which are really the same). In this case, every point on the line will be

a solution to the system. The system is called *dependent*.

That all means that a system of two linear equations in two unknowns has either no solutions, exactly one solution, or infinitely many solutions.

One method of solving a system is graphical. Carefully graph the two equations, and estimate the coordinates of the points where the graphs cross. However, there are more precise methods for dealing with two linear equations in two unknowns, and the graphical method is of no real practical value in this case.

The method of substitution: Consider the system $\begin{cases} 2x + 3y = 11 \\ 9x - 2y = 3 \end{cases}$. The plan is to solve the first equation for x (or y would be OK too), and then plug that information into the second equation. Here are the details:

Solving the first equation for x gives:

$$x = -\frac{3}{2}y + \frac{11}{2}$$

Substituting that into the second equation:

$$9\left(-\frac{3}{2}y + \frac{11}{2}\right) - 2y = 3$$

and so:

$$-\frac{27}{2}y + \frac{99}{2} - 2y = 3$$

in other words:

$$-\frac{31}{2}y = -\frac{93}{2} \qquad \text{which means} \qquad y = 3$$

Now that we know $y = 3$ we can put that information into $x = -\frac{3}{2}y + \frac{11}{2}$, and we get $x = -\frac{3}{2} \cdot 3 + \frac{11}{2} = 1$. So the solution is $(1, 3)$.

Problems to Submit

4.5.C: 76, 82

4.6.A: 24, 26

4.7.A: 12, 22

5.1.A: 2, 12

5.1.B: 36, 44

LESSON 5

More on Systems of Equations

READ:

Sections 5.2.A, 5.3.A,B, and 5.4.A

INSTRUCTIONAL NOTES:

Besides the technique learned in the last lesson, there is another method for solving a system of two equations in two unknowns.

Method 2) <u>Addition</u>. Consider the same system as in the last lesson: $\begin{cases} 2x + 3y = 11 \\ 9x - 2y = 3 \end{cases}$. The two equations will be added, the goal being to eliminate one of the unknowns. Before adding the equations, it is often necessary to multiply each of the equations by some number. The plan is to multiply by numbers that will make the coefficients of x (or y if that is easier) the negatives of each other. In that case, when the equations are added, the x terms will cancel. When you multiply an equation by a constant, be sure to remember to multiply **both sides** by the constant, so that the new equation is equivalent to the original. Here is that same example done by the method of addition.

First, multiply the top equation by 9 to get

$$18x + 27y = 99$$

Next, multiply the bottom equation by -2 to get

$$-18x + 4y = -6$$

Now add the two new equations, noting the x terms will cancel:

$$31y = 93 \qquad \text{and so} \qquad y = 3 \qquad \text{as before.}$$

Now that we know the value of y we can plug $y = 3$ into either of the original equations, and learn that $x = 1$, and so, once again, the solution to the system is $(1, 3)$.

You should practice both the method of substitution and the method of addition, and then pick which one you find easier to use. Any system can be solved with either of the two methods.

Now it's time for a few more application problems. First, let's repeat the important suggestions of Lesson 2 concerning solving word problems:

1) If possible, draw and label a diagram. If the problem concerns a rectangle, draw a rectangle and label the items in the rectangle that are known and the ones that are asked for. If the problem concerns a car on a trip from city A to city B make a dot for A and one for B, and a line joining them to represent the road. Label the diagram with information from the problem, such as direction of the car or mileage between the cities. Not all problems can be associated with a diagram, but do not skip this step if a diagram might be at all helpful.

2) Introduce a letter to stand for the unknown quantity to be found and describe the quantity in words. So if the problem asks for the speed of the car, a good start might be to write *Let* $s =$ *the speed of the car.* If the problem is to find the width of a rectangle, perhaps a good place to begin (after drawing a diagram of course) is to write *Let* $w =$ *the width of the rectangle.*

The textbook breaks word problems down into a number of different types: number problems, geometric problems, motion problems, and so on. Those are really artificial classifications. All word problems are really the same, and all are solved in the same general way - only the details differ from one problem to the next. The basic plan for solving all word problems is: 1) draw and label a diagram using information from the problem; 2) introduce a letter to represent the unknown to be found and describe the unknown in words; 3) use the information in the problem to write down an equation (or maybe a system of equations) involving the unknown; and 4) solve the equation(s). Finally, it is always a wise idea to check the solution to see if it really works. If not, try to see where the error was made: was the diagram drawn incorrectly?, was an incorrect equation written down?, was an error made when solving the equation?

Be sure to study the examples in the text thoroughly before trying the Problems to Submit for the section on applied problems, and remember that these problems might be pretty challenging. Take your time and think about what you are doing.

Certain types of problems involve the solution of a combination of several linear inequalities of the sort discussed in Lesson 4. The inequalities are considered to be combined with the word *and.* To solve a combination of several linear inequalities, solve each one individually as in the last lesson. Since they are joined by *and*, the intersection of the two solution sets is needed.

Problems to Submit

5.2.A: 16, 36

5.3.A: 6, 10

5.3.B: 16, 20

5.4.A: 8, 14

LESSON 6

Multiplying and Dividing Rational Expressions

READ:

Sections 8.1.A,B,C

INSTRUCTIONAL NOTES:

Recall that a rational number is a quotient of two integers (with the denominator not equal to 0). A **rational expression** is a quotient of two polynomials (with the denominator not equal to 0). Doing arithmetic with rational expressions is almost identical to doing arithmetic with rational numbers. For example, just as rational numbers can be reduced by canceling common factors, we can reduce rational expressions by first factoring the numerator and denominator, and then canceling common factors. Be sure only common *factors* are canceled. The following computation is **WRONG**:

$$\textbf{WRONG} \quad \frac{3x+5}{x^2+5} = \frac{3x}{x^2} \quad \textbf{WRONG}$$

where the 5 has been canceled from the numerator and the denominator. The reason this is wrong is that 5 is not a factor of the numerator and denominator, rather it is a *term* in the numerator and denominator. Terms cannot be canceled, only factors can be canceled! If you want to be convinced that the canceling above is not correct, compare the two sides of the proposed identity when a particular value is plugged in for x. For example, plugging in $x = 1$, the left side becomes $\frac{8}{6}$ or $\frac{4}{3}$, but the right side is $\frac{3}{1}$ or 3. Certainly then these two expressions are not equal. **Do not cancel terms!**

Multiplying and dividing rational functions is pretty simple. In fact, the algebra is nearly identical to the corresponding operations with rational numbers. The only real difficulty comes in those problems where you need to factor the numerator and denominator before it becomes clear what can be canceled. If you need to review the techniques of factoring (which are covered in Math 100), they are discussed in Chapter 7 of the text.

Just to jog your memory a bit, here are a few facts about factoring you will need to apply in this lesson.

To **factor** an integer means to express the integer as the product of two or more other integers. As an example, 30 can be factored as $2 \cdot 15$, or as $5 \cdot 6$, or even as $2 \cdot 3 \cdot 5$. Some integers can be factored in really only one way using positive integers. For example, $23 = 1 \cdot 23$, and that is the only possible factoring of 23, except for $23 \cdot 1$, obtained by reversing the order of the two factors.

An integer bigger than 1 that has only one possible such factoring (ignoring the order in which the factors are written) is called a **prime number**. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19,$ and 23. To factor an integer completely means to express it as a product of primes. So $60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$ is the complete factoring of 60.

To factor a polynomial means to express the polynomial as a product of two or more other polynomials. Factoring is just the reverse of multiplying polynomials together. In a factoring problem, we are told what the product is, and we have to determine what polynomials were multiplied together to produce that product. Generally speaking, factoring polynomials, just as multiplying polynomials, is an application of the distributive law: $ab + ac = a(b + c)$.

Let's warm up with an example using integers. Notice that each term of the sum $36 - 72 + 120$ has a factor of 12 in it. In fact, $36 - 72 + 120 = 12 \cdot 3 + 12 \cdot (-6) + 12 \cdot 10 = 12 \cdot (3 - 6 + 10)$. We have factored out the common factor of 12 using the distributive law. Now consider the same sort of operation with polynomials. In the polynomial $36x^2y^7 - 18x^3y^4 + 60x^5y^3$, notice that in each term there is a factor of 6 and also each term contains a factor of x^2 and also a factor of y^3. In fact

$36x^2y^7 - 18x^3y^4 + 60x^5y^3 = (6x^2y^3) \cdot (6y^4) + (6x^2y^3) \cdot (-3xy) + (6x^2y^3) \cdot (10x^3)$.

Now using the distributive law, the common factor of $6x^2y^3$ can be factored out giving

$36x^2y^7 - 18x^3y^4 + 60x^5y^3 = (6x^2y^3)(6y^4 - 3xy + 10x^3)$.

Certain trinomials (in other words, polynomials with three terms) have factorings that are very useful in solving certain types of problems. One identity mentioned in Chapter 7 is $(a+b)(a-b) = a^2 - b^2$. Written in reverse, that says $a^2 - b^2 = (a+b)(a-b)$, which says, in plain English, that the difference of two squared quantities can always be factored as the sum of the two quantities times the difference of the two quantities. So if we want to factor $x^2 - 9$, then, noting that $x^2 - 9 = x^2 - 3^2$, the above identity says $x^2 - 9 = (x + 3)(x - 3)$. In that example, the a of the identity was played by x and the b by 3. But the a and b can be played by any sorts of expressions as the following examples illustrate.

1) $16x^2 - 25y^2 = (4x)^2 - (5y)^2 = (4x + 5y)(4x - 5y)$

2) $16x^2y^6 - 100a^6b^{10} = (4xy^3)^2 - (10a^3b^5)^2 = (4xy^3 + 10a^3b^5)(4xy^3 - 10a^3b^5)$

3) $(x + 2y)^2 - 16 = (x + 2y)^2 - 4^4 = (x + 2y + 4)(x + 2y - 4)$

The other two identities you may recall, but now written in reverse, are

$$a^2 + 2ab + b^2 = (a + b)^2 \text{ and } a^2 - 2ab + b^2 = (a - b)^2$$

Using these we can factor an expression such as $x^2 + 6x + 9$ if we recognize it as $x^2 + 2 \cdot 3 \cdot x + 3^2 = (x + 3)^2$, using the first identity above.

There are two factoring patterns worth remembering:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \text{ and } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

As an example of the second one, let's factor $8x^3y^6 + y^9$. Begin by writing the given expression in the form of the left side of the identity above, and then write out the appropriate stuff for the right side:

$8x^3y^6 + z^9 = (2xy^2)^3 + (z^3)^3 = (2xy^2 + z^3)\left((2xy^2)^2 - (2xy^2)(z^3) + (z^3)^2\right)$

So $8x^3y^6 + z^9 = (2xy^2 + z^3)(4x^2y^4 - 2xy^2z^3 + z^6)$.

The identities given above are not enough to factor all types of polynomials. Rather, they are special patterns that occur frequently enough to merit special recognition. Factoring more general polynomials such as $x^2 + 5x + 6$ relies on the time-tested method of *trial-and-error*. The idea goes as follows:

If we are to write $x^2 + 5x + 6$ in factored form as $(x + a)(x + b)$ then it must be that $a \cdot b = 6$, and so the only real choices, at least if we want to stick with integers, are $(x + 1)(x + 6)$ and $(x + 2)(x + 3)$. Testing the first one, we see it multiplies out to $x^2 + 7x + 6$ and that does not give the cross product term we want. But the second one does give $x^2 + 5x + 6$, and so the factoring is $x^2 + 5x + 6 = (x + 2)(x + 3)$. That is a simple example, but it exhibits the method: guess all the possible factorings by looking at the trinomial, and then test each factoring until you find the one that works. You will discover that with practice, the number of trials and errors will decrease and that you will be able to do a lot of the steps mentally. If you exhaust the list of possible factorings and none of them worked, that means the trinomial cannot be factored. In that case, it is called a **prime** polynomial.

Considering the problem of factoring the integer 60, one possible answer would be $60 = 4 \cdot 15$. But normally when a factoring is asked for, what is wanted is a *complete* factoring. That is to say, a factoring in which the factors that appear cannot be further factored. In the case of 60, that would be $60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$. In other words, 60 has been factored into a product of primes.

The story is the same for factoring polynomials. A given polynomial should be factored into prime polynomials. Here are some examples:

$$2x^2 + 16x + 32 = 2(x^2 + 8x + 16) = 2(x + 4)^2$$

$$x^4 + 5x^2 - 6 = (x^2)^2 + 5x^2 - 6 = (x^2 + 6)(x^2 - 1) = (x^2 + 6)(x + 1)(x - 1)$$

Factoring is a much more difficult process than multiplying. It is almost always harder to take things apart than it is to mix them together. Think about how easy it is to pour milk into coffee, but how difficult it would be to get it back out again. Multiplying is like pouring milk in, factoring is like getting it back out again.

Problems to Submit

8.1.A: 14, 16, 26

8.1.B: 36, 44

8.1.C: 60, 66

If you have not yet submitted the *Request for Examination* form, you should do so now.

LESSON 7

Examination I

About the Examination

The examination will cover all the assigned sections in lessons 1 through 6. To prepare for the examination go over the homework problems you submitted for grading and read over the *Chapter Review* pages appearing at the end of each chapter. Then do the *Chapter Tests* as final preparation. The problems on the examination will be similar to the ones that you have been doing for homework and that appear in the end-of-chapter tests. There are twenty problems on the test. There will be no surprises.

For any assigned problems that seem to cause you difficulty, read over the appropriate sections in the text and practice more problems of that type.

There is a **two and one half hour** time limit for the test, but it probably won't take you that long if you are well prepared. **No** books, **no** notes, and so on, are allowed during the examination. **Calculators** are not needed and **will not be allowed** during the examination. Show all your steps for solving problems on the test so I can give you partial credit in case you make some small oversight. Arrange your work in a neat and logical fashion so the steps are easy to follow.

If you have not yet submitted the *Request for Examination* form, you should do so now.

LESSON 8

Adding and Subtracting Rational Expressions

READ:

Sections 8.2.A,B

INSTRUCTIONAL NOTES:

Multiplication and division of rational expressions is not too difficult; however, addition and subtraction are definitely more challenging. In fact, the same is true for rational numbers. It is easy to multiply $\frac{2}{3}$ and $\frac{7}{5}$ to get $\frac{14}{15}$, and it is also easy to divide $\frac{2}{3}$ by $\frac{7}{5}$, since that only involves inverting the second rational number and multiplying to get $\frac{10}{21}$.

To add (or subtract) $\frac{2}{3}$ and $\frac{7}{5}$ however, we will first need to determine the least common denominator, in this case 15. Then the two fractions are replaced by equivalent fractions with that new denominator, and finally, the new fractions are added by adding the numerators (and keeping the common denominator).

$$\frac{2}{3} + \frac{7}{5} = \frac{10}{15} + \frac{21}{15} = \frac{31}{15}$$

and

$$\frac{2}{3} - \frac{7}{5} = \frac{10}{15} - \frac{21}{15} = -\frac{11}{15}$$

The story is the same with rational expressions. To add two rational expressions, determine the least common denominator. You will probably have to factor the denominators to do that. Next, express the two rational expressions as equivalent rational expressions, both using the least common denominator. Now the two rational expressions can be added by adding the two numerators and keeping the common denominator. And they can be subtracted by subtracting numerators and keeping the common denominator.

Here is an example of the process: Let's add

$$\frac{x}{x^2 - 4} + \frac{6}{x^2 - 3x - 10}$$

Just looking at the denominators for a moment, we see $x^2 - 4 = (x+2)(x-2)$, and $x^2 - 3x - 10 = (x-5)(x+2)$. So the least common denominator will be $(x+2)(x-2)(x-5)$. That is the simplest expression that both the original denominators will divide. So,

$$\begin{aligned}
\frac{x}{x^2-4} + \frac{6}{x^2-3x-10} &= \frac{x}{(x+2)(x-2)} + \frac{6}{(x-5)(x+2)} \\
&= \frac{x}{(x+2)(x-2)} \cdot \frac{(x-5)}{(x-5)} + \frac{6}{(x-5)(x+2)} \cdot \frac{(x-2)}{(x-2)} \\
&= \frac{x^2-5x}{(x+2)(x-2)(x-5)} + \frac{6x-12}{(x+2)(x-2)(x-5)} \\
&= \frac{x^2+x-12}{(x+2)(x-2)(x-5)}
\end{aligned}$$

Make sure you use the least common denominator when adding and subtracting rational expressions. While it is true that the addition and subtraction is easy whenever the two denominators are the same, the number and complexity of the computations will be much greater if the selected denominator is not the least common denominator. As a numerical example, go back to $\frac{2}{3} + \frac{7}{5}$. We could use the common denominator 75 to do this addition. In fact,

$$\frac{2}{3} + \frac{7}{5} = \frac{50}{75} + \frac{105}{75} = \frac{155}{75}$$

But that answer is not in lowest terms, and so to get the simplest form of the answer, we would need to cancel the common factor of 5 to get $\frac{31}{15}$. The same sort of messiness will occur if you don't get the least common denominator when adding or subtracting rational expressions. All the expressions will be longer and more complicated, and, in the end, you will have to find common factors to cancel to put the expression in lowest terms.

Problems to Submit

8.2.A: 4, 20, 24

8.2.B: 28, 34, 44, 52, 62

LESSON 9

Complex Fractions and Rational Equations

READ:

Sections 8.3.A, 8.4.A, 8.5.A,B, and 8.6.A

INSTRUCTIONAL NOTES:

A complex fraction is a fraction in which either the numerator or the denominator (or both) are themselves rational expressions. Although they might look intimidating, complex fractions are actually not difficult to simplify. The hardest part is just keeping the work organized.

There are two different methods of simplifying complex fractions. One approach is to work with the numerator and denominator of the fraction separately. Each of these will be just like a problem practiced in the last few sections. Begin by just ignoring the denominator and reducing the numerator to a single fraction. After that is finished, ignore the numerator and combine the terms of the denominator into a single fraction. You should now have something that looks like

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

Where the a, b, c, and d might be pretty complicated, but each is just a single polynomial. To finish the simplification, all that is needed is to invert the denominator, multiply, and cancel as much as possible to put the answer in lowest terms.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

The text describes an alternate method. It can often arrive at the simplification in fewer steps than the method described above. The plan is to multiply the top and the bottom by a quantity that will cancel all the denominators. An example will make the method easy to understand. Let's simplify

$$\frac{x + 3 - \frac{18}{2x+1}}{x - \frac{6}{2x+1}}$$

If we multiply top and bottom by $2x+1$, the denominators of the $\frac{18}{2x+1}$ and $\frac{6}{2x+1}$ will cancel, and we will have an easy expression remaining to be simplified. Here are the details:

$$\begin{aligned}
\frac{x+3-\frac{18}{2x+1}}{x-\frac{6}{2x+1}} &= \frac{x+3-\frac{18}{2x+1}}{x-\frac{6}{2x+1}} \cdot \frac{2x+1}{2x+1} \\
&= \frac{(x+3)(2x+1)-18}{x(2x+1)-6} \\
&= \frac{2x^2+7x-15}{2x^2+x-6} \\
&= \frac{(x+5)(2x-3)}{(x+2)(2x-3)} \\
&= \frac{x+5}{x+2}
\end{aligned}$$

Reducing complex fractions is the first step in solving equations and proportions involving rational expressions.

$$\frac{1}{x-2} - \frac{2}{x+3} = \frac{11}{x^2+x-6}$$

and

$$\frac{x-3}{x+2} = 3 - \frac{1-2x}{x+2}$$

are examples of equations involving rational expressions. The easiest way to solve such equations is to determine the least common denominator (LCD) of the rational expressions appearing in the equations and then to multiply both sides of the equation by the LCD. After canceling as much as possible in each term, there will be no more rational expressions remaining and the equation to solve will be a polynomial of the sort practiced in earlier lessons. Here are the two examples above worked out in detail.

In the first example, the denominators are $x-2$, $x+3$ and $x^2+x-6=(x-2)(x+3)$, so the LCD is $(x-2)(x+3)$. Now both sides of the equation are multiplied by that LCD.

$$(x-2)(x+3)\left(\frac{1}{x-2}-\frac{2}{x+3}\right)=(x-2)(x+3)\left(\frac{11}{x^2+x-6}\right)$$

$$(x-2)(x+3)\left(\frac{1}{x-2}\right)-(x-2)(x+3)\left(\frac{2}{x+3}\right)=(x-2)(x+3)\left(\frac{11}{(x-2)(x+3)}\right)$$

$$(x+3)-2(x-2)=11$$

$$-x+7=11$$

$$x=-4$$

For the second problem, the LCD is $x+2$, and so the calculations look like:

$$(x+2)\left(\frac{x-3}{x+2}\right)=(x+2)\left(3-\frac{1-2x}{x+2}\right)$$

$$x-3=3(x+2)-(1-2x)$$

$$x-3=3x+6-1+2x$$

$$-8=4x$$

$$x=-2$$

The principle applied above to solve these two equations was: both sides of an equation can be multiplied by the same non-zero quantity without changing the solutions to the equation. When we applied that method in earlier lessons, the multiplier was always a specific number which was definitely not zero. But in the examples above, the multiplier involved an unknown and, for all we could tell, we might be multiplying by zero. That means that the new equation we produce might have solutions that do not work in the original equation. Such solutions are called **extraneous** solutions. **Whenever you carry out an operation that might introduce extraneous solutions, you must check the alleged solutions in the original equation to see if they are actually solutions to that problem**.

In the first example above, plugging $x = -4$ into the original equation, the result is

$$\frac{1}{-4-2} - \frac{2}{-4+3} = \frac{11}{(-4)^2 - 4 - 6}$$
$$-\frac{1}{6} + 2 = \frac{11}{6}$$

which is correct. So $x = -4$ is the solution to the equation.

However, when the possible solution $x = -2$ is plugged into the second equation, the result is

$$\frac{-2-3}{-2+2} = 3 - \frac{1-2(-2)}{-2+2}$$

and the division by zero gives a meaningless equation. We conclude that $x = -2$ is not a solution to the original equation; it is an extraneous solution. So, in this case, the equation has no solutions at all.

Problems to Submit

8.3.A: 10, 18, 32

8.4.A: 18, 30

8.5.A: 10, 16

8.5.B: 26

8.6.A: 20, 38

LESSON 10

Equations Involving Rational Expressions

Rational Exponents

READ:

Sections 8.7.A,B and 9.1.A,B,C

INSTRUCTIONAL NOTES:

Equations involving rational expressions pop up in a lot of applications. Pay special attention to examples 1 and 2 in section 8.6 of the text. In particular, notice how well the solutions to the problems are organized. An unknown (variable) is introduced to represent the quantity that is being sought. The variable is described in words. A diagram is drawn if possible. The information in the problem is expressed in terms of the variable and, finally, an equation involving the variable is written down. After that, the methods of this lesson are applied to solve the equation. These problems are not easy, but if you are patient and careful, you will get them.

A rational exponent is an exponent that is a positive integer, zero, a negative integer, a positive fraction, or a negative fraction. You already have studied the rules for working with integer exponents in earlier courses. Here is a quick review just to refresh your memory (and you can also check out the material in section 1.2 if you want more review).

The symbol 5^3 is shorthand for $5 \cdot 5 \cdot 5 = 125$, and is read *five to the third power*. In general, if a is any number and n is a positive integer, then a^n stands for $a \cdot a \cdot a \cdots a$, where a appears as a factor n times. The expression a^n is read *a to the n^{th} power*. The quantity a is called the *base* and n is called the *exponent* in that expression. Note that a^2 is usually read *a squared* and a^3 is read as *a cubed*. The reasons for these words are geometric: a square with sides of length a has area a^2, and a cube of side length a has volume a^3.

The most common error when working with exponents is confusing the two quantities $(-5)^2$ and -5^2. In the first, the quantity-5 is to be multiplied by itself, so that $(-5)^2 = (-5)(-5) = 25$. The second expression stands for the **negative of** 5^2, and so $-5^2 = -(5)(5) = -25$. Be sure this distinction is clear.

Let's do a quick review of the basic laws of algebra. The arithmetic operations (addition, subtraction, multiplication, division, and exponentiation) always operate with two numbers at a time. If three numbers are to be combined using these operations, it is necessary to indicate the order in which the numbers are to be combined. For example, without more explanation, it is not clear what is meant by $2+3\cdot 5$. There are two reasonable values that can be assigned to that expression: one is produced by adding 2 to 3 and multiplying the total by 5 to get 25. The other possibility is to first multiply 3 by 5 and add 2 to that for a total of 17.

One way to distinguish these two possibilities is to use parentheses to make it clear which interpretation is wanted: the first one above would be written as $(2+3)\cdot 5$, while the second would be written $2+(3\cdot 5)$. The rule is that work is completed inside the parentheses first. While it is always possible to make the meaning clear for a complicated expression by using enough parentheses (or brackets $[\cdots]$ or braces $\{\cdots\}$), it soon becomes very difficult to read, and the quantities look very cluttered and confusing. To reduce the need for such groupings, people have agreed on certain rules for the order of doing arithmetic. The rules are

1) If there are any groupings, do all arithmetic within a group first.

2) Within a group, do all exponentiation operations in order, left to right.

3) Next, within a group, do all multiplications and divisions in order, left to right.

4) Finally, within a group, do all additions and subtractions in order, left to right.

Applying these rules, the agreed upon steps to find the value of $2+3\cdot 5$ are first to use rule 3, to get $2+15$, and then use rule 4 to get 17.

There are certain laws that the arithmetic operations obey. For example, the order in which two numbers are added has no effect on the total: $3+5=8$, and also $5+3=8$, or, in general, for any two numbers a and b, $a+b=b+a$. This is called the **commutative property** of addition (since the two numbers can *commute* or *move around* without changing the total). The various laws are given names.

For all numbers a, b, and c,

1) (The commutative properties for addition and multiplication) $a+b=b+a$ and $ab=ba$.

2) (The associative properties of addition and multiplication) $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c$.

3) (The distributive property) $a(b + c) = ab + ac$.

4) (The identity properties) $a + 0 = a$ and $1 \cdot a = a$.

5) (The inverse properties) $a + (-a) = 0$ and if $a \neq 0$, $a \cdot \left(\frac{1}{a}\right) = 1$.

6) (The zero multiplication property) $a \cdot 0 = 0$.

The properties of real numbers are used to simplify complicated expressions. A **term** is a single number, or a single number multiplied by variables raised to powers. Examples of terms are -12, $3x^2y^5$, and t. On the other hand $4a^3b - 7x$ is made up of two separate terms: $4a^3b$ and $-7x$. The **coefficient** of a term is the number part of the term. For example, the coefficient of -12 is -12, the coefficient of $3x^2y^5$ is 3, and the coefficient of t is 1. Note that $1 \cdot t = t$, so that a coefficient of 1 in usually omitted.

Two terms are said to be **like terms** if they contain the same variables with the same exponents. So $-4x^3yz^7$ and $110x^3yz^7$ are like terms, but $-4x^3yz^7$ and $5x^3yz^6$ are not since z has exponent 7 in the first term, but exponent 6 in the second. The reason like terms are important is that they can be added easily using the distributive property. If we think of x^3yz^7 as *apples*, then $-4x^3yz^7 + 110x^3yz^7$ means 110 apples minus 4 apples, which is surely 106 apples, so that $-4x^3yz^7 + 110x^3yz^7 = 106x^3yz^7$. This is really an application of the distributive law, as explained in the text. Notice that trying to add two unlike terms, say $-4x^3yz^7 + 110x^3yz^6$, is like trying to add apples and oranges. General rule: do not add unlike terms.

Now the important thing to keep in mind is that the rules for working with fractional exponents are exactly the same, so there really isn't anything new to learn. For example,

$$a^{\frac{1}{2}} \cdot a^{\frac{2}{3}} = a^{\frac{1}{2}+\frac{2}{3}} = a^{\frac{5}{6}}$$

which follows the rule of adding exponents we learned for integer exponents.

We know that x^5 means *multiply x times itself* 5 *times*, but what is the meaning of the expression $x^{\frac{1}{2}}$? To answer that question, we make the following definition. Suppose n is a positive integer. A number a is called an n^{th} **root of** b provided $a^n = b$. So 2 is a 6^{th} root of 64 since $2^6 = 64$. Also -2 is another 6^{th} of 64 since $(-2)^6 = 64$. That means 64 has two different 6^{th} roots. The number -27 has one 3^{rd} root, namely -3, since $(-3)^3 = -27$.

A special name is introduced for 3^{rd} roots: they are called **cube roots**. So -3 is the cube root of

-27 Also, a special word is introduced for 2^{nd} roots: they are called **square roots**. So 16 has two square roots, namely 4 and -4, since $4^2 = 16$ and $(-4)^2 = 16$.

There are three important facts to keep in mind about roots:

1) A positive number such as 16 above will have two square roots, 4 and -4, and it also has two 4^{th} roots, namely 2 and -2 (since $2^4 = 16$) and $(-2)^4 = 16$. In general, a positive number will have two square roots, two 4^{th} roots, two 6^{th} roots, two 8^{th} roots, and so on. And, in each case, the two roots will be negatives of each other.

2) A negative number cannot have a square root, or a 4^{th} root, or a 6^{th}, or an 8^{th} root, and so on, since any number raised to such an even power will never be negative. For example, there are no square roots of -4, since we cannot square a number and get the negative answer -4.

3) A negative number will have one 3^{rd} root, one 5^{th} root, one 7^{th} root, and so on, and in each case, that root will be a negative number. For example, -1000 has one cube root, namely -10, since $(-10)^3 = -1000$.

In the case of an even root of a positive number as described in paragraph (1) above, the positive root is called the **principal** root. So we say 2 is the principal square root of 4, and that 10 is the principal cube root of 1000 and that 2 is the principal 6^{th} root of 64.

Now we are prepared to give meaning to fractional exponents. The expression $a^{\frac{1}{n}}$ stands for the n^{th} root of a. If a is positive and n is even, then $a^{\frac{1}{n}}$ means the principal n^{th} root of a. So, for example $(-8)^{\frac{1}{3}} = -2$, $64^{\frac{1}{6}} = 2$, and $(-4)^{\frac{1}{2}}$ has no meaning.

Another symbol for n^{th} roots is $\sqrt[n]{}$. So the above three examples can also be written as $\sqrt[3]{-8} = -2$, $\sqrt[6]{64} = 2$, and $\sqrt[2]{-4}$ has no meaning. When writing square roots with the $\sqrt[2]{}$ symbol, the 2 is normally dropped, so that we would write $\sqrt{25} = 5$.

A common **error** is to write $\sqrt{25} = \pm 5$, meaning that there are two values for $\sqrt{25}$, namely 5 and -5. That is **wrong**. There is only one value for $\sqrt{25}$ and that is the principal value 5. Notice that $-\sqrt{25} = -(\sqrt{25}) = -5$.

More complicated fractional exponents can be handled by using the rules of exponents. For example: $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$.

Negative fractional exponents are dealt with just like negative integer exponents. Usually, the easiest way to compute a^{-r} is to invert a and raise the result to the power r. For example

$\left(\frac{8}{27}\right)^{-\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} = \frac{27^{\frac{1}{3}}}{8^{\frac{1}{3}}} = \frac{3}{2}$.

It is sometimes more convenient to work with expressions involving the radical sign $\sqrt[n]{}$ than to write out fractional exponents. Since $\sqrt[n]{a}$ is exactly the same thing as $a^{\frac{1}{n}}$, it does not really matter whether you use radical signs or fractional exponents. The only difference is that an expression written with radicals is probably a bit easier to read.

Problems to Submit

8.7.A: 8, 18

8.7.B: 24, 34

9.1.A: 40, 64

9.1.B: 90, 110

9.1.C: 122, 128

LESSON 11

Simplifying Radical Expressions

READ:

Sections 9.2.A,B,C,D

INSTRUCTIONAL NOTES:

There is a little bit of vocabulary needed to discuss the rules for manipulating radicals. In the expression $\sqrt[n]{a}$, the n is called the **index** of the radical and the a is called the **radicand**. The basic rules for working with radicals are:

1) Multiply and divide radicals only if they have the same index. Thus $\sqrt[3]{4} \cdot \sqrt[3]{11} = \sqrt[3]{44}$, but there in no nice way to simplify $\sqrt[3]{4} \cdot \sqrt[5]{11}$.

2) Add and subtract radicals only if they have the same index and radicand. For example, $\sqrt{3} + 2\sqrt[3]{7} + 5\sqrt{3} - 4\sqrt[3]{7} = 6\sqrt{3} - 2\sqrt[3]{7}$. Notice that there is no nice way to combine the terms in $\sqrt{3} + \sqrt[5]{3}$, because the indexes are different. Likewise, $\sqrt{7} + \sqrt{11}$ cannot be simplified since the radicand are different.

The rules of exponents can be used to simplify expressions containing radicals. Normally, when simplifying a radical the goal is to make the radicand as simple as possible. Notice that when simplifying radicals, it is usually assumed that any variables represent positive real numbers so that all the roots will make sense. As an example of simplification, $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. In this case, the radicand included a perfect square factor of 25 which we took outside the radical sign, where it turned into a 5. Another example of the same sort of process is $\sqrt[3]{24x^5y^{11}z^3} = \sqrt[3]{8x^3y^9z^3 \cdot 3x^2y^2} = \sqrt[3]{8x^3y^9z^3} \cdot \sqrt[3]{3x^2y^2} = 2xy^3z\sqrt[3]{3x^2y^2}$. In this case, we took as many perfect cubes outside the radical as we could.

Another type of problem involving radicals is called **rationalizing the denominator**. When a fraction involves radicals in the denominator, it is often convenient to modify the fraction to eliminate those radicals. They are eliminated by multiplying the numerator and denominator of the fraction by the same quantity, and by selecting that quantity so that the radicals in the denominator disappear. An example will illustrate the ideas.

Consider the quotient $\frac{6}{\sqrt{14}}$. The radical in the denominator can be eliminated if it is multiplied by $\sqrt{14}$. In fact

$$\frac{6}{\sqrt{14}} = \frac{6}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}} = \frac{6\sqrt{14}}{14} = \frac{3\sqrt{14}}{7}$$

In the third step we used the observation that $\sqrt{14} \cdot \sqrt{14} = 14$. In general, $\sqrt{a} \cdot \sqrt{a} = a$ for any number a bigger than or equal to zero.

Numbers of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called **conjugates** of each other. Notice that these are all square roots and that the radicands are the same in both expressions. The only difference is that the first expression is the sum of the two radicals, while the second expression is the difference. Conjugate numbers play an important role in simplifying certain types of fractions involving radicals. The reason is that when a number and its conjugate are multiplied together, the product is very simple. In fact, the identity $(x + y)(x - y) = x^2 - y^2$ tells us that $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$. So, for example, the product of $\sqrt{7} + \sqrt{5}$ and its conjugate is $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5}) = 7 - 5 = 2$.

The conjugate numbers can be used to simplify certain quotients of numbers involving square roots. The idea is to multiply the numerator and denominator of the expression by the conjugate of the denominator. As observed above, that operation will produce a simple denominator. Here is an example:

$$\frac{2 + 3\sqrt{5}}{3 - \sqrt{5}} = \frac{2 + 3\sqrt{5}}{3 - \sqrt{5}} \cdot \frac{3 + \sqrt{5}}{3 + \sqrt{5}} = \frac{6 + 2\sqrt{5} + 9\sqrt{5} + 3 \cdot 5}{9 - 5} = \frac{21 + 11\sqrt{5}}{4}$$

Problems to Submit

9.2.A: 6, 12

9.2.B: 28, 38

9.2.C: 54, 72

9.2.D: 92, 106

LESSON 12

Radical Equations

READ:

Sections 9.3.A,B

INSTRUCTIONAL NOTES:

Solving equations involving radicals or fractional exponents can be pretty tricky. Notice that the equation $x^3 = 27$ has exactly one solution, namely $x = 3$, and also the equation $x^5 = -32$ has exactly one solution, namely $x = -2$. In general, an equation of the form $x^n = a$ (where a is some number) will have exactly one real solution *whenever n is an odd integer*. That solution will be $\sqrt[n]{a}$. On the other hand, the equation $x^2 = 100$ has two solutions, namely $x = 10$ and $x = -10$, or, written more compactly, $x = \pm 10$. Likewise, $x^6 = 64$ has two real solutions, $x = \pm 2$. In general, the equation $x^n = a$, where a is a **positive number**, will have exactly two real solutions, $x = \pm\sqrt[n]{a}$ *whenever n is an even integer*. Finally, an equation like $x^2 = -9$ has no real solution since an even power of a real number x cannot be negative. In this case, the solutions are the complex numbers $x = \pm 3i$ which are introduced in the last section of chapter 9. That topic will be covered after the next exam.

As an example of these ideas in action, consider the equation $9(x-2)^2 = 25$. To solve for x, let's first multiply both sides by $\frac{1}{9}$ to get $(x-2)^2 = \frac{25}{9}$. Applying the principle described in the last paragraph, we see $x - 2 = \pm\sqrt{\frac{25}{9}} = \pm\frac{5}{3}$. That means $x = 2 \pm \frac{5}{3}$, and so there are two solutions to the equation: $x = \frac{11}{3}$ and $x = \frac{1}{3}$.

When an equation involves radicals, a good idea is to raise both sides of the equation to a power that will eliminate the radicals. The computations can get rather sticky, so be sure you write out every step carefully. Here is an example of the process in action: let's solve $\sqrt{x+20} - \sqrt{x} = 2$. Since the equation involves square roots, we will square both sides to get rid of the radicals. It is often a good idea to have the square roots on opposite sides of the equation before squaring. It makes the algebra a little easier. The computations go as follows:

$$\left(\sqrt{x+20}\right)^2 = \left(2+\sqrt{x}\right)^2$$
$$x + 20 = 4 + 4\sqrt{x} + x$$
$$16 = 4\sqrt{x}$$
$$4 = \sqrt{x}$$
$$x = 16$$

So we found a solution. But wait! There is a danger: **when both sides of an equation are raised to an even power, extraneous solutions may be introduced**. In this case, checking the alleged solution in the original equation, we discover the $x = 16$ does work since $\sqrt{16+20} - \sqrt{16} = 6 - 4 = 2$ is correct. Always check for extraneous solutions whenever you raise both sides of an equation to an even power.

For the word problems, be sure to organize your work as described in earlier lessons.

Problems to Submit

9.3.A: 4, 8, 16, 18

9.3.B: 22, 26

You may submit the *Request for Examination* form with this lesson. See the next lesson for information about the examination.

LESSON 13

Examination II

About the examination

The examination will cover all the assigned sections in lessons 8 through 12. As usual, to prepare for the examination go over the homework problems you submitted for grading and read over the *Chapter Review* pages appearing at the end of each chapter. Then do the *Chapter Tests* as final preparation.

The problems on the examination will be similar to the ones that you have been doing for homework and that appear in the end-of-chapter tests. There are twenty problems on the test. There will be no surprises.

For any assigned problems that seem to cause you difficulty, read over the appropriate sections in the text and practice more problems of that type.

There is a **two and one half hour** time limit for the test, but it probably won't take you that long if you are well prepared. **No** books, **no** notes, and so on, are allowed during the examination. **Calculators** are not needed and **will not be allowed** during the examination. Show all your steps for solving problems on the test so I can give you partial credit in case you make some small oversight. Arrange your work in a neat and logical fashion so the steps are easy to follow.

If you have not yet submitted the *Request for Examination* form, you should do so now.

LESSON 14

Complex Numbers

READ:

Sections 9.4.A,B,C,D

INSTRUCTIONAL NOTES:

The equation $x^2 = -25$ has no real solutions since squaring a real number cannot produce a negative answer. To make such equations solvable, new types of numbers called **complex numbers** are introduced. We will define a new quantity i, called the **imaginary unit**, and declare that $i^2 = -1$. In other words, $i = \sqrt{-1}$. The imaginary unit is treated just like a real number, except that whenever i^2 appears in a computation, it can be replaced by -1. The imaginary unit is combined with real numbers to produce numbers like $3 + 5i$, $-2 + \sqrt{7}i$, $32i$, and in general, $a + bi$, to form the complex numbers.

How about the equation $x^2 = -25$? We can now write $x = \pm\sqrt{-25} = \pm\sqrt{(25)(-1)} = \pm\sqrt{25} \cdot \sqrt{-1} = \pm 5i$. So the equation has two solutions, both of them complex numbers.

It is possible to add, subtract, multiply, and divide complex numbers and the results will be other complex numbers. The arithmetic with complex numbers works pretty much the same as arithmetic with real numbers, except that whenever i^2 appears in a computation, it is replaced by -1. Here are a few examples to give you the idea.

$(3 + 7i) + (4 - 2i) = 7 + 5i$

$(3 + 7i) - (4 - 2i) = -1 + 9i$

$(3 + 7i)(4 - 2i) = 12 - 6i + 28i - 14i^2 = 12 - 6i + 28i - 14(-1) = 12 - 6i + 28i + 14 = 26 + 22i$

The process of simplifying the quotient of two complex numbers is a little more complicated. The trick is almost like the process of rationalizing denominators involving radicals practiced in the last lesson. First a definition: The **complex conjugate** of the number $a + bi$ is the number $a - bi$. In plain English, to form the conjugate of a complex number, change the sign of the coefficient of i. The conjugate of $3 + 7i$ is $3 - 7i$. The conjugate of $4 - 9i$ is $4 + 9i$.

The product of a complex number and its conjugate turns out to be very simple: $(a + bi)(a - bi) =$

$a^2+abi-abi-bi^2 = a^2-b^2(-1) = a^2+b^2$. So, for example, we can write $(4+3i)(4-3i) = 4^2+3^2 = 25$. Notice that the product of a complex number and its conjugate is always a real number. Now to compute the quotient of two complex numbers, multiply the numerator and denominator by the conjugate of the denominator and then combine like terms and cancel common factors if possible. An example of the procedure:

$$\frac{2+3i}{3+5i} = \frac{2+3i}{3+5i} \cdot \frac{3-5i}{3-5i} = \frac{(2+3i)(3-5i)}{(3+5i)(3-5i)} = \frac{6-10i+9i-15i^2}{3^2+5^2} = \frac{21-i}{34} = \frac{21}{34} - \frac{1}{34}i$$

Problems to Submit

9.4.A: 8, 12

9.4.B: 16, 20

9.4.C: 32, 36, 40

9.4.D: 44, 56, 58

LESSON 15

Quadratic Equations

READ:

Sections 10.1.A,B,C and 10.2.A

INSTRUCTIONAL NOTES:

One reason for learning to factor polynomials is that factoring can be used to solve equations concerning polynomials. The fact that makes factoring useful for solving equations is the **zero factor property** of numbers which says that **if the product of two numbers is** 0**, then at least one of the factors is** 0. Expressing the same thing symbolically: If $ab = 0$, then either $a = 0$ or $b = 0$.

To see how useful that observation is, consider the problem of solving the equation $x^2 + 5x + 6 = 0$. If we recognize that the left side can be factored as $(x+2)(x+3)$, then we see the problem is equivalent to solving the equation $(x+2)(x+3) = 0$, and the zero factor property tells us that is the same as $x + 2 = 0$ or $x + 3 = 0$. Now, those two equations are simple to solve. The first says $x = -2$, while the second says $x = -3$. So we conclude the equation $x^2 + 5x + 6 = 0$ has two solutions, namely $x = -2$ and $x = -3$. Sure enough, if you plug those two values in for x in the left side of the equation you do get 0 as desired.

The general procedure for using factoring and the zero factor property for solving is to, first, write the equation in the form $f(x) = 0$, and then, second, factor $f(x)$, and third, set each factor equal to 0 which will produce several simpler equations, and, finally, solve those simpler equations.

Here is a more involved example of the same process. Let's solve $x^4 + 5x^2 - 6 = 0$. Factoring the left side completely, we get $(x^2+6)(x+1)(x-1) = 0$. Using the zero factor property, we know that either $x^2 + 6 = 0$ or $x + 1 = 0$ or $x - 1 = 0$. Now the equation $x^2 + 6 = 0$ solutions $x = \pm\sqrt{6}i$, while the other two equations have solutions $x = -1$ and $x = 1$. Thus the equation $x^4 + 5x^2 - 6 = 0$ has two real solutions, $x = \pm 1$, and two non-real solutions $x = \pm\sqrt{6}i$.

An error that is sometimes made is replacing the 0 in the zero factor property by other numbers. For example, suppose we wanted to solve the equation $x^2 + 5x + 6 = 1$. Factoring the left side we get $(x+2)(x+3) = 1$, which is correct. The error sometimes made is to now say: either $x+2 = 1$ or $x + 3 = 1$, a sort of 1 *factoring property*. Solving those two equations tells us that $x = -1$ and $x = -2$ are the solutions to the original equation. But checking these two values by plugging them in for x in the original equation, tells us that neither one really works. What went wrong? Of course the answer is that there is no such thing as the 1 *factor property*. It can happen that the product of two numbers can be equal to 1 without either of the factors equal to 1. For example, $2 \cdot \frac{1}{2} = 1$, but certainly neither 2 nor $\frac{1}{2}$ is equal to 1. The zero factor property is a special property belonging only to 0, so do not try to use it for other numbers in place of 0!

An equation of the form $ax^2+bx+c = 0$ is called a **quadratic** equation. Here the a, b, and c are some specific numbers, and $a \neq 0$. There is another technique, besides factoring, for solving quadratic equations that can be used even when factoring is very difficult. For example, $x^2 + 5x + 6 = 0$ is equivalent to $(x + 2)(x + 3) = 0$, which has solutions $x = -2, -3$. But the method of factoring is not always so easy to apply. It is difficult to see how to factor the left side of $x^2 + x - 3 = 0$ for example; however, there is a routine way of solving all such quadratic equations without having to figure out how to factor the quadratic polynomial.

First a little vocabulary. A quadratic expression such as $x^2 + 2x + 1$ is called a **perfect square** because it is the square of expression $x+1$. That is, $(x+1)^2 = x^2+2x+1$. On the other hand $2x^2+6$ is not a perfect square since there is no expression of the form $ax+b$ for which $(ax+b)^2 = 2x^2+6$. That is easy to see since $2x^2 + 6$ has no cross product term. Likewise $x^2 + 6x$ is not a perfect square.

The second example above is interesting. $x^2 + 6x$ is not a perfect square, but it is only a little bit short! If there were only a 9 added to the end, then $x^2 + 6x + 9 = (x + 3)^2$, as is easy to check. The process of adding a real number to an expression of the form $ax^2 + bx$ to make it a perfect square is called **completing the square**. The easiest examples to work with are those where the coefficient of the x^2 term is 1, so let's start with those. Suppose we wanted to add a number to $x^2 + 8x$ to make it a perfect square. So we want to find a number c and a number d so that $x^2 + 8x + c = (x + d)^2$. In other words, we need $x^2 + 8x + c = x^2 + 2dx + d^2$. Now if you look at the coefficient of x on each side of that equation you see $8 = 2d$, and so $d = 4$. Since the equation also says $c = d^2$, we conclude the number to add is $c = d^2 = 4^2 = 16$. And, sure enough, we have $x^2 + 8x + 16 = (x + 4)^2$, so we have completed the square. If you look at the work we just did, you can see the general rule: *To complete the square for $x^2 + bx$, the number to add is $\left(\frac{b}{2}\right)^2$.* To complete the square for $x^2 + 18x$, add $9^2 = 81$, To complete the square for $x^2 - 20x$ add $(-10)^2 = 100$, to complete the square for $x^2 + 5x$ add $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

If the coefficient of x^2 is not 1, then completing the square is a little more work, but still not difficult. An example or two will show what to do. To complete the square for $2x^2 + 12x$, begin by factoring the 2 from each term to get $2(x^2 + 6x)$. Now, complete the square inside the parentheses by adding 9. $2(x^2+6x+9) = 2(x+3)^2$. Here is an important observation. When we added 9 inside the parentheses, we actually added 18 to the original expression, since the 9 inside the parentheses gets multiplied by the 2 on the outside. Here's one more. Let's complete the square for $3x^2 - 5x$. We first factor out the 3 to get $3(x^2 - \frac{5}{3}x)$. Next we take half the coefficient of x (which is $-\frac{5}{6}$), square that, to get $\frac{25}{36}$, and add that inside the parentheses, to get $3(x^2 - \frac{5}{3}x + \frac{25}{36}) = 3(x + \frac{5}{6})^2$. How much did we really add that time? Well, we added $\frac{25}{36}$ inside the parentheses, but everything inside the parentheses is multiplied by 3, so we actually added $3 \cdot \frac{35}{36} = \frac{25}{12}$ to the original expression.

Now let's solve some quadratic equations by completing the square. First a really simple one where the solutions are obvious. Solve $x^2 = 4$. Here the solutions are clearly $x = \pm 2$. Here is a slightly harder one. Solve $(x - 7)^2 = 9$. Well, if the square of $x - 7$ is 9, then $x - 7$ must be either 3 or -3, so $x - 7 = \pm 3$. That means $x = 7 \pm 3 = 10, 4$. Here's one more just a bit more complicated. Solve $2(x - 2)^2 = -5$. Let's first divide by 2 to get $(x - 2)^2 = -\frac{5}{2}$. We see that equation means $x - 2 = \pm\sqrt{\frac{5}{2}}i$. Thus $x = 2 \pm \sqrt{\frac{5}{2}}i$ are the two solutions to the equation.

In those examples, it was not necessary to complete the square. In fact, they each already involved a perfect square. Now consider this problem. Solve $x^2 + 6x - 3 = 0$. The left side is not a perfect square, but we can make it one. Here are the steps:

$$
\begin{aligned}
x^2 + 6x - 3 &= 0 \\
x^2 + 6x &= 3 \\
x^2 + 6x + 9 &= 3 + 9 \\
(x + 3)^2 &= 12 \\
x + 3 &= \pm\sqrt{12} = \pm 2\sqrt{3} \\
x &= -3 \pm 2\sqrt{3}
\end{aligned}
$$

In the first step, 3 was added to both sides of the equation so the left side would look like the sort of thing for which we know how to complete the square. Then 9 was added to both sides of the equation. We know we can always add the same thing to both sides of an equation without changing the solutions. Next the left side was written as a perfect square, and we combined numbers on the right side of the equation. Now we have a simple sort of problem to solve: $(x+3)^2 = 12$. Writing down the solution to that, and then subtracting 3 from both sides of the equation, we end up with the solutions.

Here's an even messier example: Solve $2x^2 - x - 15 = 0$. Here are the steps:

$$\begin{aligned}
2x^2 - x - 15 &= 0 \\
2x^2 - x &= 15 \\
2(x^2 - \frac{1}{2}x) &= 15 \\
x^2 - \frac{1}{2}x &= \frac{15}{2} \\
x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{15}{2} + \frac{1}{16} &= \frac{121}{16} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{121}{16} \\
x - \frac{1}{4} = \pm\sqrt{\frac{121}{16}} &= \pm\frac{11}{4} \\
x &= \frac{1}{4} \pm \frac{11}{4} \\
x &= 3, -\frac{5}{2}
\end{aligned}$$

Be sure you understand what happened at each step. That solution looks fairly complicated, but the good news is that that is about as hard as these sorts of problems can get, so if you understand that one, you can do any quadratic by the method of completing the square.

Problems to Submit

10.1.A: 20, 24, 34

10.1.B: 58

10.1.C: 88, 92, 102

10.2.A: 8, 24, 44

LESSON 16

The Quadratic Formula

READ:

Sections 10.3.A and 10.5.A

INSTRUCTIONAL NOTES:

After you have solved a dozen or so problems by completing the square, it will become apparent that you are doing the exact same seven or eight steps every time. In fact, it is possible to carry out the steps of solving by completing the square once and for all starting with $ax^2 + bx + c = 0$. The details are provided on the first page of section 10.3. The result is a formula that gives the solutions to $ax^2 + bx + c = 0$ in terms of a, b, and c. The **quadratic formula** says the solutions will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula amounts to an automatic completion of the square. It saves you the trouble of doing the tedious algebra yourself. For the equation above, $2x^2 - x - 15 = 0$, the $a = 2$, $b = -1$, and $c = -15$, and so the quadratic formula says the solutions are

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-15)}}{2 \cdot 2} = \frac{1 \pm \sqrt{121}}{4} = \frac{1 \pm 11}{4} = 3, -\frac{5}{2}$$

just as in the last lesson, but with lots less labor, and lots less chance for errors.

Sometimes it is hard to see how to factor a quadratic. It might take quite a few trial-and-error attempts before you stumble over the correct factoring of $6x^2+31x-60$. The quadratic formula can be used to find the factors without all the trial and error. Solving the equation $6x^2 + 31x - 60 = 0$ using the quadratic formula, the results are $x = -\frac{20}{3}$ and $x = \frac{3}{2}$. The first of these is the same as $3x+20 = 0$ and the second is the same as $2x-3 = 0$. The factors of $6x^2+31x-60$ are $3x+20$ and $2x-3$. Checking, we find $6x^2 + 31x - 60 = (3x + 20)(2x - 3)$ is correct. You should always do this last little check because it might happen that there is a coefficient missing on the right side which you will have to supply. For example, if you use this trick to factor $2x^2 + 10x + 12$, you will find the quadratic formula gives you solutions $x = -3$ and $x = -2$ to the equation $2x^2 + 10x + 12 = 0$, and as above, we would conclude $x+3$ and $x+2$ are factors of $2x^2+10x+12$. But when these two factors are multiplied together, the result is x^2+5x+6. Looking at that, and what we want as the product, we can see the correct factoring is $2x^2+10x+12 = 2(x+2)(x+3)$. We needed to adjust the left side by putting in a constant factor of 2.

Suppose we wanted to solve the inequality $x^2-x-6>0$. Notice that $x=5$ is one solution, since $5^2-5-6=14$ and that is greater than 0. But $x=1$ is not a solution since $1^2-1-6=-6$ and -6 is not greater than 0. One more: $x=-3$ is a solution since $(-3)^2-(-3)-6=6$, and that is greater than 0. How can we determine all the x values for which the inequality is true? The easiest way is to first factor the left side of the inequality to get $(x-3)(x+2)$, and note the values of x when this expression equals 0. Those values are $x=3$ and $x=-2$. If the points $x=-2$ and $x=3$ are plotted on the real line, they separate the x-axis into three intervals. Draw a diagram!

The intervals are $x<-2$, $-2<x<3$, and $x>3$. A convenient test value is selected from each interval and plugged into the inequality. For example, we might select $x=-5$ in the left most interval. Plugging that in to the inequality, we get $(-5)^2-(-5)-6>$ which is true. The important fact is that if one number in the interval passes the test, then all the numbers in the interval will pass the test. That means that all the numbers in $(-\infty,-2)$ will be part of the solution we are looking for. In the interval $-2<x<3$, we might select the test value $x=0$. It fails the test. So none of the numbers in $-2<x<3$ will work. Finally, testing say $x=5$ from the last interval, we see $5^2-5-6>0$ is true, so $x>3$ is also part of the solution. So the solution to $x^2-x-6>0$ consists of the two intervals $x<-2$ and $x>3$.

The general procedure for solving these quadratic inequalities is to first rewrite the problem so that it has just a 0 on one side of the inequality. For example, to solve $x^2+5x\le 11$, rewrite it as $x^2+5x-11\le 0$. Next, solve the equation $x^2+5x-11=0$. The solutions to this equation will separate the x-axis into a number of intervals. To see if an interval is part of the solution or not, select one test value from the interval and plug it into the inequality to see if it works.

The same process works with rational inequalities. For example, let's solve $\dfrac{5x+9}{x-3}\ge 2$. To get a 0 on the right side, we will subtract 2 from both sides of the inequality.

$$\frac{5x+9}{x-3} \geq 2$$

$$\frac{5x+9}{x-3} - 2 \geq 0$$

$$\frac{3x+15}{x-3} \geq 0$$

Now we determine the x values which make the numerator or denominator equal to 0. They are $x = -5$ and $x = 3$ in this example. These values split the x-axis into three intervals: $x < -5$, $-5 < x < 3$, and $x > 3$. Testing one number from each interval, we see that the solution to the original inequality is $x < -5$ together with $x > 3$. Actually, in this example, if $x = -5$ then the inequality is also correct, so in fact the solution is $x \leq -5$ or $x > 3$. Notice that we cannot include $x = 3$ in the solution since when $x = 3$ the denominator of the rational expression in the problem will be 0, and division by 0 is undefined.

A function of the form $f(x) = mx + b$ (where m and b are particular numbers) is called a **linear function**. The graph of a linear function is a straight line of slope m, and y-intercept b. Those two pieces of information make it easy to sketch the graph of a linear function without having to build a table of values for the function.

Problems to Submit

10.3.A: 2, 14, 20, 24, 26

10.5.A: 4, 6, 12, 26

LESSON 17

More Quadratic Equations

READ:

Sections 10.6.A

INSTRUCTIONAL NOTES:

Word problems leading to quadratic equations are discussed in section 10.6.A. Be sure to go back to lesson 2 and read the general advice concerning solving word problems. As the problems become more involved, it becomes even more important to organize your work in a logical fashion.

Some of the word problems will need a few facts from geometry:

1) The area of a rectangle is its height times its width.

2) The area of a triangle is $\frac{1}{2}$ the base times the height.

3) The volume of a rectangular box is the height times the width times the length.

4) In a right triangle the square of the hypotenuse equals the sum of the squares of the two legs.

A more complete list of facts can be found inside the cover of your text. More details and examples appear in Chapter 3 of the text.

Problems to Submit

10.6.A: 2, 4, 6, 10, 14

LESSON 18

Parabolas

READ:

Sections 11.1.A,B,C,D

INSTRUCTIONAL NOTES:

Functions of the form $f(x) = ax^2 + bx + c$ are called **quadratic functions**. Here a, b, and c are numbers (and $a \neq 0$). Graphs of quadratic functions are parabolas, opening either up or down.

When graphing a parabola, as for graphing any function, it is possible to simply plug in many choices for x, compute the corresponding value for y, and plot the points (x, y). However, this *plot-a-million-points* method of drawing graphs should be a method of last resort. For example, when graphing a function of the form $f(x) = 3x + 2$, we could plot a million points, or we could notice that f is a linear function and so the graph will be a straight line - that means we need only plot two points. An even more sophisticated observation is that the line has y-intercept 2 and slope 3, and those facts lets us quickly sketch the graph of f, essentially without plotting any points at all.

The idea is the same for quadratic functions. Instead of plotting many points, we look for a few features of the function that allow us to quickly sketch the graph with next to none of that tedious point plotting. Since we know the basic shape is going to be a parabola, we can get a decent graph by determining just two facts: (1) does the parabola open up or down? and (2) where is the vertex (the lowest or highest point) of the parabola? With these two questions answered, we can draw a quick sketch of the parabola. If we need a little more precise graph, we can determine the x and y intercepts of the graph (always easy to do). So when graphing a parabola, there is no need for a large table of (x, y) points on the graph.

Here are the facts you need to learn about the graph of a quadratic function $f(x) = ax^2 + bx + c$.

1) If a is positive the parabola opens upwards. If a is negative, the parabola opens downwards.

2) The vertex occurs at $x = -\frac{b}{2a}$. Plugging that x value into the function f will give the y coordinate of the vertex.

For example, the parabola $f(x) = 4x^2 - x + 3$ opens upwards (since 4 is positive) and the vertex occurs at $x = -\frac{-1}{2\cdot 4} = \frac{1}{8}$. That means the y coordinate of the vertex is $f(\frac{1}{8}) = 4(\frac{1}{8})^2 - \frac{1}{8} + 3 = \frac{47}{16}$, so the vertex is at $(\frac{1}{8}, \frac{47}{16})$. That's enough information to draw a rough sketch of the graph of f.

The only trouble with this short cut to graphing a parabola is trying to remember the formula for the x coordinate of the vertex of the parabola. Luckily, you don't have to memorize it. Instead, you can produce that x coordinate automatically. Here's how: Start with $f(x) = ax^2 + bx + c$, and complete the square on the right. The result will look something like $f(x) = a(x-d)^2 + e$ (where d and e are some numbers). Then the vertex will have x coordinate d. Free bonus: the y coordinate of the vertex is e. Isn't that neat!

Here's an example.

$$\begin{aligned} f(x) = 4x^2 - x + 3 &= 4\left(x^2 - \frac{1}{4}x\right) + 3 \\ &= 4\left(x^2 - \frac{1}{4}x + \frac{1}{64}\right) + 3 - 4\cdot\frac{1}{64} \\ &= 4\left(x - \frac{1}{8}\right) + \frac{47}{16} \end{aligned}$$

And so we see the vertex is at $\left(\frac{1}{8}, \frac{47}{16}\right)$, just as before, but this time, we found it *automatically.*

If you want to provide a little more detail for the graph, you can plot the intercepts. To find the y-intercept, simply set $x = 0$. For the example above, $f(x) = 4x^2 - x + 3$, the y-intercept is $(0, 3)$. The x-intercepts (if there are any at all) are a little more trouble: Set $y = 0$ and solve for x. For the example $f(x) = 4x^2 - x + 3$, we would solve $4x^2 - x + 3 = 0$. Using the quadratic formula gives $x = \frac{1\pm\sqrt{1-48}}{2\cdot 4}$. These values are imaginary, so this parabola has no x-intercepts (which would also be clear if you sketch the graph). On the other hand, the parabola with equation $f(x) = x^2 - 3x - 10$ has two x-intercepts. They are found by solving $0 = x^2 - 3x - 10 = (x-5)(x+2)$. So the x-intercepts are $(5, 0)$ and $(-2, 0)$.

Problems to Submit

11.1.A: 10, 20, 24

11.1.B: 36, 52

11.1.C: 74, 78

11.1:D: 90, 92

You may submit the *Request for Examination* form with this lesson. See the next lesson for information about the examination.

LESSON 19

Examination III

About the examination

The examination will cover all the assigned sections in lessons 14 through 18. As usual, to prepare for the examination go over the homework problems you submitted for grading and read over the *Chapter Review* pages appearing at the end of each chapter. Then do the *Chapter Tests* as final preparation.

The problems on the examination will be similar to the ones that you have been doing for homework and that appear in the end-of-chapter tests. There are sixteen problems on the test. There will be no surprises.

For any assigned problems that seem to cause you difficulty, read over the appropriate sections in the text and practice more problems of that type.

There is a **two and one half hour** time limit for the test, but it probably won't take you that long if you are well prepared. **No** books, **no** notes, and so on, are allowed during the examination. **Calculators** are not needed and **will not be allowed** during the examination. Show all your steps for solving problems on the test so I can give you partial credit in case you make some small oversight. Arrange your work in a neat and logical fashion so the steps are easy to follow.

If you have not yet submitted the *Request for Examination* form, you should do so now.