Math 352, Handout I:
Convergence of Series Solutions

1 Overview

In Example 2 on page 610 of our textbook ([4]), the authors work out a series solution to a certain heat flow problem. In this handout, I would like to explain why the series solution actually converges to a solution. By studying this handout, you may learn how to convince yourself that the series solutions of certain other equations actually converge to solutions. **This handout is *not* required reading for this course.**

The solution that the authors come up with is the following:

$$u(x,t) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} e^{-2(2k+1)^2 t} \sin(2k+1)x. \tag{1}$$

A moment's consideration shows that each term of this series is a solution to the given partial differential equation (Equation (9) on page 610) which satisfies the desired boundary conditions (Equation (10) on page 610). Thus if we can differentiate the series term-by-term, then it follows that the series gives a solution to (9). Note that the discussion on page 602 of the text gives an example in which you *cannot* differentiate a series term-by-term, so we have some work to do to show that we can differentiate the authors' solution term-by-term. But before we do that, we note a few things.

Note first that since each term of the series in (1) is zero at $x = 0$ and $x = \pi$, it follows that $u(0,t) = u(\pi,t) = 0$. The series solution thus satisfies the boundary conditions (10) in the text. Also note that if $t = 0$ in (1), above, we find that

$$u(x,0) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin(2k+1)x. \tag{2}$$

According to Example 1 on page 610 of our textbook, this is the Fourier sine series for the initial condition function given in equation (11) of Example 2. As in Example 1, let $f(x)$ denote this initial condition function. Since the odd 2π-periodic extension of $f(x)$ is continuous, and since its derivative is piecewise continuous, it follows from Theorem 2 on page 600 that (2), above, converges to $f(x)$ for all x in the interval $[0,\pi]$. In other words, the series solution (1), above, not only satisfies the desired boundary conditions, but it also satisfies the desired initial conditions.

2 Working with the Partial Differential Equation

We now turn to the problem of showing that the series (1), above, actually satisfies the desired partial differential equation. As we do this, we will assume that $0 < x < \pi$ and $0 < t$. Later on, we will show that the series converges to a function which is continuous on $\{(x,t)|0 \le x \le \pi, 0 \le t\}$. Once we have done these things, it will be clear that (1), above, gives a solution to the original heat flow problem from Example 2.

We begin by defining the concept of uniform convergence of a sequence of functions. The textbook discusses uniform convergence on page 601. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions defined on in interval $[a,b]$, and let $f(x)$ be any function defined on $[a,b]$. We say that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges to f on $[a,b]$ *uniformly* if for any $\epsilon > 0$, there exists an number N such that for any $n \ge N$ and for any x in $[a,b]$, we have $|f_n(x) - f(x)| < \epsilon$. The key point here is that the same N works for all x in $[a,b]$. Figure 10.8 on page 601 of the text illustrates this.

We will also need the Weierstrass M-test, which we quote (in adapted form) from page 214 of [2]. This test also appears in many other books on advanced calculus and real analysis.

Theorem 1: The Weierstrass M-Test. Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of functions defined on a set E. Suppose that, for each nonnegative integer n, there is a number M_n such that

$$|f_n(x)| \le M_n \quad \text{for every } x \in E.$$

If the series $\sum_{n=0}^{\infty} M_n$ converges, then the series of functions $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on E.

To see what it means to say that the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on E, let $S_n(x) = \sum_{i=0}^{n} f_i(x)$ for all $x \in E$. (You may assume that E is an interval $[a,b]$). When we say that $\sum_{n=0}^{\infty} f_n$ converges uniformly on E, we simply mean that there is a function f defined on E such that the sequence $\{S_n(x)\}_{n=0}^{\infty}$ converges to f uniformly on E.

To show that (1) satisfies the original differential equation, we will need one more tool, namely the following theorem, which we quote (in adapted form) from page 208 of [2]:

Theorem 2: Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of functions that are differentiable on the interval $[a,b]$. Suppose that

(i) there is a point $x_0 \in [a,b]$ where $\{f_n(x_0)\}_{n=0}^{\infty}$ converges, and

(ii) $\{f_n'(x)\}_{n=0}^{\infty}$ converges uniformly on $[a,b]$.

Then

(a) $\{f_n\}_{n=0}^{\infty}$ converges uniformly to a function f on $[a, b]$, and

(b) $f'(x) = \lim_{n\to\infty} f_n'(x)$ on (a, b).

Now return to (1), above. Let $S_n(x, t)$ denote the n'th partial sum of this series. Thus

$$S_n(x,t) = \frac{4}{\pi}\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} e^{-2(2k+1)^2 t} \sin(2k+1)x.$$

Moreover,

$$\frac{\partial S_n}{\partial t}(x,t) = \frac{4}{\pi}\sum_{k=0}^{n} -2(-1)^k e^{-2(2k+1)^2 t} \sin(2k+1)x.$$

We will apply Theorem 2 to $\{S_n\}_{n=0}^{\infty}$ and $\{\frac{\partial S_n}{\partial t}\}_{n=0}^{\infty}$. Fix constants x and t_0 such that $0 < x < \pi$ and $0 < t_0$. You should think of t_0 as being a very small positive constant. Also, think of $S_n(x,t)$ and $\frac{\partial S_n}{\partial t}(x,t)$ as functions of the single variable t, for any $t \in \mathbb{R}$, and let $b > t_0$ be given. In the Weierstrass M-test, let $M_k = \frac{4}{\pi} 2e^{-2(2k+1)^2 t_0}$. It follows from the Ratio Test that $\sum_{k=0}^{\infty} M_k$ converges. Thus by the Weierstrass M-test, $\{\frac{\partial S_n}{\partial t}(x,t)\}_{n=0}^{\infty}$ converges uniformly on $[t_0, b]$. On the other hand, the Ratio Test shows that $\sum_{n=0}^{\infty} \frac{4}{\pi}\frac{1}{(2k+1)^2} e^{-2(2k+1)^2 t_0}$ converges. It therefore follows from the Comparison Test that the right-hand side of (1) and hence also $\{S_n(x,t)\}_{n=0}^{\infty}$ converge for any t in $[t_0, b]$. So we see that the seqeuence $\{S_n(x,t)\}_{n=0}^{\infty}$ satisfies both of the hypotheses of Theorem 2. Thus by Part (b) of Theorem 2, $\frac{\partial u}{\partial t}(x,t)$ exists on (t_0, b). It also follows from Part (b) of Theorem 2 that for all t in (t_0, b), $\{\frac{\partial S_n}{\partial t}(x,t)\}_{n=0}^{\infty}$ converges to $\frac{\partial u}{\partial t}(x,t)$. But $\{\frac{\partial S_n}{\partial t}(x,t)\}_{n=0}^{\infty}$ is just the sequence of partial sums of the series we obtain when be take the term-by-term partial derivative with respect to t of the series in (1). We can therefore compute the desired partial derivative of the series in (1) by doing term-by-term differentiation.

A similar argument shows that we can compute the second-order partial derivative with respect to x of the series in (1) by doing term-by-term differentiation. One main thing that makes all of this work is the factor $e^{-2(2k+1)^2 t}$. As long at $0 < t_0 \leq t$, this allows us to use the Ratio Test and the Comparison Test to show that the series in question converges. This is true no matter how many times we differentiate the terms of the series.

Since (for $t_0 < t$) we may differentiate (1) term-by-term, it is now easy to see that (1) gives a solution to the differential equation in Example 2 on page 610 of our textbook.

You may still be concerned about the t_0 in the above argument. But if you are working at any point (x,t) with $0 < x < \pi$ and $0 < t$, there is always a t_0 such that $0 < t_0 < t$. Thus the series satisfies the differential equation at (x,t).

Finally, we would like to know that (1) converges to a function which is *continuous* on the set of all (x, t) such that $0 \leq x \leq \pi$ and $0 \leq t$. (Note that the inequalities are not strict here.) By "continuous" here, we mean continuous in the sense of Calculus III. To show that the series in (1) converges to a continuous function, we need two more theorems. First, we will need the following modified version of the Weierstrass M-test:

Theorem 3: Let $\sum_{n=0}^{\infty} M_n$ be a convergent series of nonnegative numbers. Consider a series $\sum_{n=0}^{\infty} a_n(x,t)$ whose terms $a_n(x,t)$ satisfy $|a_n(x,t)| \leq M_n$ for all (x,t) in some subset D of $\mathbb{R}^2$. Then $\sum_{n=0}^{\infty} a_n(x,t)$ converges uniformly on D.

We never defined uniform convergence for a sequence of functions of more than one variable, but you can probably figure out what the definition should be. The proof of Theorem 3 is an exercise in advanced calculus. I do not have a good reference for it. We will also need the following simplified version of a theorem from page 178 of [1]:

Theorem 4: Let $\{f_n(x,t)\}_{n=0}^{\infty}$ be a sequence of continuous real-valued functions defined on a subset D of $\mathbb{R}^2$. Suppose also that $\{f_n(x,t)\}_{n=0}^{\infty}$ converges uniformly to a function $f(x,t)$. Then $f(x,t)$ is continuous.

We may now show that the series in (1) converges to a continuous function. We apply Theorem 3 with $M_0 = \frac{4}{\pi}$ and $M_k = \frac{4}{\pi k^2}$ for all $k > 0$. It follows from the theorem that the partial sums of the series in (1) converge uniformly on $\{(x,t)|0 \leq x \leq \pi, 0 \leq t\}$. Since these partial sums are continuous functions, it follows from Theorem 4 that the series in (1) converges to a continuous function $\{(x,t)|0 \leq x \leq \pi, 0 \leq t\}$.

Let me know if you have any questions. If you are interested in this material, you may wish to consider taking our Advanced Calculus sequence (Math 431-432). This course sequence usually covers sequences of functions.

References

[1] Hoffman, K. *Analysis in Euclidean Space.* Englewood Cliffs, New Jersey: Prentice-Hall, 1975.

[2] Kirkwood, J.R. *An Introduction to Analysis.* Boston: PWS-Kent, 1989.

[3] Munkres, J. *Topology: A First Course.* Englewood Cliffs, New Jersey: Prentice-Hall, 1975.

[4] Nagle, R.K., Saff, E.B., and Snider, A.D. *Fundamentals of Differential Equations and Boundary Value Problems, Fourth Edition.* Boston: Addison-Wesley, 2004.