Math 208, Handout IX: Review Exercises for Midterm III

To prepare for Midterm III, you may wish to study these problems, the problems in Handout VIII, the assigned homework problems, the book, and your lecture notes. The exam will cover chapters 16 through 24, but not Chapter 20. You may *not* use calculators during the exam.

1. Find the best possible big-oh estimate for each of the following functions:

   (a) $\log_2(2^{3n}+1)$ (b) $(n+50+\frac{1}{n})^3$ (c) $\sqrt{9n^6+3n^5+\frac{1}{n}}$ (d) $2^{n+50}+n^3$

2. Briefly explain what you would have to do to prove that 998759 is a prime number.

3. In each case, do the following: (1) Use the Euclidean algorithm to compute $\gcd(a,b)$, and (2) write $\gcd(a,b)$ as a linear combination of a and b. In (2), use the method of backsubstitution. Do not use the continued fraction method.

   (a) $a=18,\ b=64$ (b) $a=42,\ b=11$ (c) $a=16,\ b=37$ (d) $a=30,\ b=72$

4. In each case, find $\gcd(a,b)$.

   (a) $a=392,\ b=0$ (b) $a=36,\ b=36$ (c) $a=24,\ b=1$ (d) $a=42,\ b=21$

5. Give an example of a pair of composite integers a and b which are relatively prime.

6. Find the prime factorization of 484000. How many positive divisors does this number have?

7. Find all of the positive divisors of the number 114.

8. Let $n\in\mathbb{Z}$ be given. What can you conclude about $\gcd(12,n)$?

9. Let integers x and y be given, and let $n=16x+72y$. If $n>0$, what is the smallest number that n can possibly be?

10. Let integers a, b, and n be given. Prove the following:

    If $n|ab$ and n and a are relatively prime, then $n|b$.

11. Prove the second theorem on page 108 of our textbook. In particular, prove the following:

    If p is prime and $p|a_1a_2\cdots a_n$, then $p|a_j$ for some $j=1,2,\cdots,n$.

    Hint: Use the theorem that you are asked to prove in Problem 10, above.

12. Let a, $b\in\mathbb{Z}$ be given, and suppose a and b are not both zero. Let $d=\gcd(a,b)$. Prove that $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

13. Let integers a and b be given, and suppose that there exist integers s and t such that $as+bt=36$. What can you conclude about $\gcd(a,b)$?

14. Design an algorithm whose input is a list of one or more positive integers and whose output is "Yes" if at least three of the integers in the list are equal to 7 and "No" if at most two of the integers are equal to 7. What is the worst case scenario efficiency function, $w(n)$, for your algorithm?

15. Consider the following algorithm. (a) Describe in words what this algorithm does. In other words, what problem does this algorithm solve? (b) What is the worst case scenario efficiency function $w(n)$?

    Input: An integer $n > 1$.

    Step 1: Set $s = \sqrt{n}$.
    Step 2: Set $i = 1$.
    Step 3: Replace i with $i + 1$.
    Step 4: If $i > s$, then output "Yes" and stop.
    Step 5: If $\lfloor \frac{n}{i} \rfloor = \frac{n}{i}$, then output "No" and stop.
    Step 6: Go to Step 3.

16. Consider the following algorithm. (a) Describe in words what this algorithm does. In other words, what problem does this algorithm solve? (b) What is the worst case scenario efficiency function $w(n)$?

    Input: Two lists of positive integers of length n, where $n \geq 1$:

    $a_1, a_2, \ldots, a_n$
    $b_1, b_2, \ldots, b_n$

    Step 1: Set $i = 1$.
    Step 2: Set $j = 1$.
    Step 3: If $\lfloor \frac{b_j}{a_i} \rfloor = \frac{b_j}{a_i}$, then output "Yes" and Stop.
    Step 4: Replace j with $j + 1$.
    Step 5: If $j \leq n$, then go to Step 3.
    Step 6: Replace i with $i + 1$.
    Step 7: If $i \leq n$, then go to Step 2.
    Step 8: Output "No" and stop.

17. Let a positive integer n be given, and suppose that $p_1, p_2, \ldots, p_n$ are distinct prime numbers. Let nonnegative integers $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ be given. Prove the following statement:

    If $i_k \neq j_k$ for some k, then $p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} \neq p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n}$.

    Remark: The idea here is that by putting different exponents on the primes, you get different numbers. This fact is important in Problem 6, above.