Math 512-513, Handout IX:
Homework Involving Isometrically Isomorphic Normed Vector Spaces

Please read the paragraph in Section 10.3 which immediately follows the proof of Proposition 7. Note how the author states that "we saw that the dual of L^p was (isometrically isomorphic to) L^q for $1 \leq p < \infty$." In this situation people will often say that the dual of L^p *is* L^q, but in fact the dual of L^p is just isometrically isomorphic to L^q. You should try to accustom yourself to thinking that the dual of L^p really *is* L^q (for $1 \leq p < \infty$), but in the back of your mind you should remember the isometric isomorphism. In this exercise, you will prove that $(L^p)^*$ is in fact isometrically isomorphic to L^q, for $1 \leq p < \infty$.

Let p be given, where $1 \leq p < \infty$, and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Let $T : L^q \to (L^p)^*$ be defined as follows. For every $g \in L^q$, let F_g be the linear functional on L^p given by integration against g. Specifically, for all $f \in L^p$, let $F_g(f) = \int_0^1 fg$. Then F_g is clearly a linear functional. By Hölder's inequality, it is clear that F_g is bounded. Thus $F_g \in (L^p)^*$. Let $Tg = F_g$. Do each of the following exercises and hand in your work by the announced due date.

1. Show that the map $T : L^q \to (L^p)^*$ is linear. Hint: if you are confused, you may wish to take a look at Proposition 3 in Section 10.2 along with its proof. It is important that you understand the vector space structure on $(L^p)^*$.

2. Show that the map $T : L^q \to (L^p)^*$ is surjective (i.e. onto). Hint: This will not be difficult if you use the correct theorem.

3. Show that the map $T : L^q \to (L^p)^*$ is injective (i.e. one-to-one). Suggestion: Let h_1 and h_2 in L^q be given. Show that $(Th_1 = Th_2) \Rightarrow (h_1 = h_2)$. To do this, suppose that $Th_1 = Th_2$, and note that $T(h_1 - h_2) = \theta$, where θ is the zero element of $(L^p)^*$. Use this fact to show that $h_1 - h_2 = 0$ a.e.

4. Show that the map $T : L^q \to (L^p)^*$ preserves norms. In other words, show that for all $g \in L^q$, $\|F_g\| = \|g\|_q$. Hint: You may wish to take a quick look at Section 6.5.

5. Consider the map $T^{-1} : (L^p)^* \to L^q$. This map is clearly a bijective (i.e. one-to-one onto) map which preserves norms. Prove that T^{-1} is linear.