Math 115, Handout 4: Test for Divisibility by 7

We have discussed techniques for determining whether or not a positive integer is divisible by 2, 3, 4, 5, 6, 8, or 9. In this handout, we discuss a test for determining whether or not a given positive integer n is divisible by 7. This test is not as easy to use as the tests we have discussed so far, but it works. We will not bother to prove that it works.

To see how the test for divisibility by 7 works, we begin by letting n be any given positive integer. We know that we can express n as follows:

$$n = a_k \times 10^k + a_{k-1} \times 10^{k-1} + \cdots + a_2 \times 10^2 + a_1 \times 10^1 + a_0 \times 10^0 \tag{1}$$

Here a_k, a_{k-1}, …, a_2, a_1, and a_0 are the digits of n. In (1), replace each of the digits a_k, a_{k-1}, …, a_2, a_1, and a_0 with the remainder you get when you divide it by 7. Another way to say this is to say that you should make the replacements according to the following chart:

Digit	0	1	2	3	4	5	6	7	8	9
Replacement	0	1	2	3	4	5	6	0	1	2

Also replace each 10 in (1) with the remainder you get when you divide 10 by 7. In other words, replace each of the 10's in (1) with 3. Leave the exponents in (1) unchanged.

Now simplify the expression that results from the above procedure. At any point in this simplification process, you may replace a number with the remainder you would get if you divided it by 7. *But never do this with any exponents.* After you finish this simplification process, the number that results is divisible by 7 if and only if n is divisible by 7.

Example 1: Let $n = 98$. Then $n = 9 \times 10 + 8$. If you make the replacements that we described, you obtain $2 \times 3 + 1 = 7$. But 7 is divisible by 7, so 98 is also divisible by 7. In particular, $98 = 49 \times 2 = 7 \times 7 \times 2 = 7 \times 14$.

Example 2: Let $n = 871$. Then $n = 8 \times 10^2 + 7 \times 10 + 1$. If we make the replacements, we obtain $1 \times 3^2 + 0 \times 3 + 1 = 9 + 1 = 10$. Since 10 is not divisible by 7 we see that 871 is *not* divisible by 7. In fact, $871 \div 7 = 124.4$.

Example 3: Let $n = 458,913$. Then

$$n = 4 \times 10^5 + 5 \times 10^4 + 8 \times 10^3 + 9 \times 10^2 + 1 \times 10 + 3$$

If we make the replacements, we obtain

$$4 \times 3^5 + 5 \times 3^4 + 1 \times 3^3 + 2 \times 3^2 + 1 \times 3 + 3$$

It is not clear whether or not this number is divisible by 7. But $3^5 = 3^2 \times 3^2 \times 3 = 9 \times 9 \times 3$, and we may replace 9 with 2. So we replace 3^5 with $2 \times 2 \times 3 = 4 \times 3 = 12$.

Similarly, $3^4 = (3^2)^2 = 9^2$, so we replace this with $2^2 = 4$.

We know that $3^3 = 9 \times 3$, and we replace this with $2 \times 3 = 6$.

If we make all of these replacements, we obtain

$$4 \times 12 + 5 \times 4 + 1 \times 6 + 2 \times 2 + 3 + 3 = 48 + 20 + 6 + 4 + 6 = 48 + 20 + 16$$

But we may replace 48, 20, and 16 with 6, 6, and 2, respectively, since these are the remainders that you get after dividing 48, 20, and 16 by 7. We finally obtain

$$6 + 6 + 2 = 14$$

Since 14 is divisible by 7, we see that 458,913 is divisible by 7. In fact, $458,913 \div 7 = 65,559$.