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Using Peer Review to Foster Proof-Writing Skills in Mathematics

I: Teaching and Learning Strategy

• The Instructional Context – Understanding Mathematical Proof

A mathematical proof is an argument, utilizing the rules of logic, that is used to establish the truth of a mathematical statement. Proofs are generally written with full attention to the use of complete sentences, paragraphs, and proper grammar. In this sense a proof is an essay which combines use of, for example, the English language and the notation utilized as part of the language of mathematics.

- The ability to read, write, and understand proofs is an essential part of what it means to be mathematically educated.
- Training in these skills comprises a prominent part of many undergraduate mathematics courses.

• The Strategy – Using Peer Review to Teach Proof Writing

By adapting the peer review model commonly used to teach college composition (c.f., [Crow, Melanie & Helmstetter, Allen, Editors (2003), *The University of North Dakota Guide to College Composition, Third Edition*, The University of North Dakota Composition Program, Department of English, Grand Forks, ND]), students learning to write proofs are asked to become involved in reading and critiquing proofs written by their peers, and in responding to the peer feedback they receive. Instructor evaluation of the work produced at each step in this process further adds to the feedback received by students as they fulfill their roles as readers and writers of proofs.

- The peer review component that has been developed adds to the usual approach to teaching proof writing by:
 - * Requiring students to submit selected proofs for review by their classmates.
 - * Involving students in reading and formally critiquing each other's proofs.
 - * Requiring the author of a proof to respond to the comments they receive from their peers.
- Students are given guidance throughout this process in at least two specific ways:
 - * As they read and critique their peers' work, students must complete a *Peer Review Worksheet* (see attached) in order to help make the review as useful as possible to the proof's author.
 - * Each student must then respond to the review they receive by completing an *Author's Response Sheet* (also attached), which helps them to articulate their reasons for responding to the review in the way that they do.
- Each student receives instructor feedback on all three aspects of their work:
 - * In addition to being reviewed by a peer, a copy of each student's original proof is read and graded by the instructor.
 - * Two copies of each review are submitted – one for the author and one for the instructor – so that the instructor can grade each student on their role as reviewer.
 - * The final draft of each proof is also graded by the instructor.
 - * Instructor feedback on each element of the complete cycle is provided after the final draft has been submitted.

II: Development of this Strategy

• How Proof Writing is Usually Taught

In proof-based mathematics courses, students are generally given a large number of examples of correct, well-written, proofs. These examples are encountered throughout the text, and frequently a large amount of class time is devoted to discussing the elements that comprise a well-written proof. Typically students are then asked to practice their proof-writing skills by constructing their own proofs throughout the term.

- A typical proof-based mathematics course focuses its attention on reading and discussing *correct* proofs.
- Student efforts are then frequently devoted to practicing and honing their ability to *write* proofs.

• Why Should We Add Anything to this Pedagogical Approach?

Since proofs are used to provide verifications of mathematical facts, it is essential that students understand these proofs in order to truly understand the mathematics involved. Thus, there are content-level reasons why students should be exposed to a large number of correct, well-written proofs. The importance of this exposure is also supported by the contention that one would expect good readers generally to be good writers. Despite these reasons, it is easy to imagine students becoming too unengaged – reading the words without understanding the sentences – when exclusively reading arguments that they know in advance to be correct. They might then be inclined to use a fact or technique when writing their own proofs that, although they would have seen it used in a similar situation before, they should realize does not apply if they had only more carefully read the proof in which they first saw it used. This argues for giving students some training in reading an argument not just to learn content, but also to read it in order to determine if in fact it provides valid reasoning.

- The emphasis placed on correct proofs is, for important reasons, a cornerstone of the training of a mathematician.
- However, when writing proofs, students also need to be able to critically evaluate their own argument in order to verify its correctness.
- This suggests that giving students opportunities to critique incorrect proofs would provide valuable experience.

• Adding a Critical Reading Component to the Class

The question now becomes where to find examples of proofs that students can not assume are correct. There is at least one example of a text, focused on teaching the basics of proof writing, that has these types of examples [Sibley, Thomas Q. (2009), *Foundations of Mathematics*, John Wiley & Sons]. However, most mathematics content areas do not seem to have any readily available resources that would suit the purpose at hand. However, regardless of the class, as long as students are being asked to write their own proofs, there will be a source of potentially incorrect proofs that are specifically tailored to that class's content area.

- Student-written proofs are occasionally incorrect, and therefore could serve as a source of material that other students would need to critically evaluate.
- Aside from this there are two very important additional benefits:
 - * Authors of proofs receive feedback on their work that they must also critically evaluate for correctness and appropriateness.
 - * Authors of proofs have an opportunity to revise their work, thereby modeling an important writing strategy that is applicable in other academic contexts.

III: Intended Student Learning Outcomes

- **Students Must Become More Active Readers of Proofs:**

It is common for mathematics instructors to advise their students to read any mathematical argument with pencil and paper at the ready, so as to be able to check the correctness of all aspects of the arguments they will encounter. However, if these arguments are known to be correct there is little motivation for taking this advice. Reading their peers' work, on the other hand, means there are no longer any assurances that the details they are reading will be correct.

- **Students Must Become More Critical Readers of Proofs:**

In every instance where a student is reading a proposed proof of a given statement, they will have already written their own argument in support of that same statement. Thus, they will have at least two examples of arguments given to support the same statement. In some cases these arguments will be essentially the same; in others they will be different – not just cosmetically, but also in their basic approach to the problem. As such, a student must critically evaluate each argument to see whether or not it is correct, even if that argument is completely different than their own.

- **Students Must Become More Refined Writers of Proofs:**

Once a student has read one of their peers' proofs and also received feedback from a peer on the proof they originally wrote, they will be in possession of three different perspectives for that problem: their original argument, the feedback received for their argument, and the argument given by one of their peers. At this stage it will be necessary for the student to evaluate the critique they have been given, and then possibly to weave the salient points from this analysis together with the two complete arguments that they would have seen.

IV: Assessment of Student Learning

- **How Have Students Responded to Doing Peer Reviews?**

After one semester of implementing this strategy, when asked, most students indicated that they thought the peer review assignments were important for helping them to learn the proof-writing process, and that the process of commenting on their peers' proofs made them more aware of how to correctly write their own proofs. Not all students, however, felt that the peer review assignments were worth the time that they required to complete.

- **Is There Any Evidence of Student Learning?**

A significant amount of student work was collected in order to assess the peer review component of the class. A brief review of this evidence reveals that in most cases students adequately fulfilled their roles as reviewers. This led to a significant number of second drafts or appropriate decisions not to revise the original proof. Furthermore, a large majority of the second drafts were improvements on the original proof. This last fact, in particular, provides strong evidence of student learning since it implies that understanding was gained as a result of having participated in the peer review process.

- There were 105 usable instances where the complete peer review cycle took place.
 - * Only rarely (in 2 out of 53 instances) did a reviewer read a correct first draft and give advice that would have resulted in an incorrect revision.
 - * Of the incorrect first drafts, 62% were recognized by the reviewer as being incorrect, and at least some useful advice for correcting the problem was given. In fact, in 35% of the reviews of incorrect first drafts, precisely the right advice for correcting the problem was given.
 - * Of those instances where a review contained some advice for the proof's author, 52% of the time a revised second draft was submitted, and 23% of the time an appropriate justification was given for not using that advice to revise the proof.

- * In those cases where an author submitted a revision, 85% of the revised proofs were noticeable improvements on the original.
- How should we interpret this evidence in terms of the intended learning outcomes?
 - * The sheer amount of feedback that students routinely gave as part of a review provides evidence that students were, at least for the peer review assignments, meeting the *active readers of proofs* outcome.
 - * Supporting the *critical readers of proofs* outcome we note that:
 - When presented with a correct proof, students were fairly adept at recognizing it as such.
 - Although less consistent in recognizing an incorrect proof, students were still usually able to at least realize something about the proof was not correct.
 - * There is strong evidence students were achieving the *refined writers of proofs* outcome:
 - Over 75% of the time authors responded appropriately to the advice they received in a review. This suggests that in most cases students were learning the content of the course well enough to accurately interpret the critique they had been given.
 - The large percentage of revisions that improved on the original strongly suggests student learning took place as a result of the peer review process.

V: The Strategy's Applicability

• Implementation Considerations

In order to successfully implement this peer review process, there are at least two important considerations: the time it takes to complete each cycle of the process and the degree to which the students are dependent upon each other for timely and accurate reviews.

- Each cycle of the peer review process takes approximately one week to complete once the first draft of a proof has been submitted.
- It is difficult to manage situations where a student does not submit their review by the due date, thereby putting the author of the proof at a disadvantage for submitting a timely revision.
- An inaccurate review may mislead a student into revising their proof in inappropriate ways.
 - * The burden of interpreting the review accurately is still put on the author of the proof.
 - * This can be mitigated by placing a significant grade penalty on inaccurate reviews.
 - * However, this is a potential pedagogical advantage since it forces students to critically evaluate the reviews they receive.
- For all of these reasons, there is a significant grading and administrative burden placed on the instructor.
 - * For any work subject to the peer review process, the grading load is approximately triple what it would otherwise be.
 - * Simply keeping all of the various drafts and reviews organized, as well as making sure the students submit all work on time, can lead to an unusual amount of administrative overhead.

• Applicability

At its heart, this strategy is simply asking students to evaluate the work of other students, and therefore could potentially be used in any type of course and in any discipline. Although, it would certainly not lend itself to student work that can be evaluated in an automatic way, such as multiple choice questions. A key to getting the most from this strategy, at least in terms of student learning, is that each stage in the cycle involves work that can only be interpreted by being very thoughtful. Some possibilities might be:

- Any situations in which students must develop an argument to support a claim.
- Situations where students are asked to provide detailed solutions to potentially complex problems, such as in many mathematics and science courses.

Peer Review Worksheet

Instructions: After reading your assigned proof, answer the following questions.

1. What is your name?
2. Who's proof did you review?
3. What is the plan of the proof? That is, is it a direct proof, a proof by contradiction, a proof by contrapositive, a proof by contradiction, or a proof by induction?
4. Is the beginning of the proof consistent with the plan of the proof? If not, what needs to be changed? Be specific!
5. Reread each sentence of the proof carefully. Is the proof entirely correct? If not, where is the *first* place that it breaks down? Mark this place with a #1 and describe here *exactly* how that problem can be corrected. Again, be specific!

6. Will the proof be entirely correct with the revision you suggested above? If not, describe in detail what else is wrong with the proof, marking each specific point with #2, #3, etc., and then providing specific suggestions here for how each of these issues can be corrected.

7. Is the end of the proof consistent with the plan of the proof? If not, what needs to be changed?

8. Be sure to make two copies of this worksheet and attach a marked-up copy of the proof you reviewed to each one.

Author's Response Sheet

Instructions: After reading the reviewer's comments, answer the following questions.

1. What is your name?
2. Who reviewed your proof?
3. Are you submitting a revised version of your proof?
4. Did the reviewer have any suggestions for revisions that you are not adopting? If so, you *must* provide a justification for why you are not adopting them. Be specific!

5. Are you making any substantive changes to your original proof that have nothing to do with the reviewer's comments? If so, explain why.

6. If you are revising your proof, make sure to attach the revised version to this worksheet.